



Archivo
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MESON THEORY
(F. Low, 1959)

Introduction

The region of validity of our theories varies with the energy
For non-relativistic quantum mechanics:

$$E \sim 10 \text{ e.v.} \quad (\text{energy})$$

$$L \sim 10^{-8} \text{ cm.} \quad (\text{distance})$$

$$\lambda \sim 10^{-5} \text{ cm.} \quad (\text{wave length})$$

For relativistic electrodynamics (quantum electrodynamics)

$$E \sim 1 \text{ Mev}$$

$$\lambda \sim 10^{-10}, 10^{-11} \text{ cm.}$$

We put in theory the mass, spin, charge of electron and spin and mass of photon. We can calculate and compare with experiment since the fine structure constant:

$$\frac{e^2}{hc} = \frac{1}{137}$$

is small. The region of validity of Q.E. is O.K if not occur before nuclear phenomena. In this later case it will be impossible tell anything about pure Q.E.

E.g. for nuclear electric field.

$$V(\Sigma) = \int \frac{e(\Sigma') d\Sigma'}{|r - \Sigma'|} \rightarrow \frac{q}{r} \text{ only for large } r$$

The charge distribution q in the deuteron gives a contribution of $\frac{2}{3} mc$ in Lamb shift.

It is expected since is reasonable that the spectra for a distribution of sources was different from a point source.

The principal wave lengths and rest energies are:

$$\text{For electron: } \frac{t_0}{mc} = 4 \times 10^{-4} \text{ cm} \quad mc^2 = \frac{1}{2} \text{ Mev.}$$

$$\text{For pion: } \frac{t_0}{mc} = 1.4 \times 10^{-13} \text{ cm} \quad mc^2 \sim 140 \text{ Mev.}$$

$$\text{For nucleon: } \frac{t_0}{mc} = 0.2 \times 10^{-14} \text{ cm} \quad mc^2 \sim 1 \text{ Bev.}$$

The quantum electrodynamics is unsatisfactory because of the need of renormalization. However any improvement of Q.E. is very difficult because it almost always leads to violation of needed requirements.

We will try, in this course, to see what is the validity of the analog theory of nuclear interactions.

Direct π -N interactions will give us more and easier information than nuclear forces.

We will be interested in low energies (~ 10 MeV) and not at 1 GeV.
The elementary or composed character of a system depend on the range of energy considered. For example in the liquid problem the atom of He is an elementary particle.

Fermi and Yang suggested that π is a bound state of N and \bar{N} .
One can not know the validity of this assumption by phenomena at energies less than the energy needed to dissociate bound state (In this case there will be a lot of binding energy).

It is also hard to find out anything about K mesons and hyperons since we need 3-times as much energy.

In this course we will study only meson-nucleon-photon phenomena
In this kind of phenomena the coupling constant is:

$$\frac{g^2}{tc} \approx 15$$

We cannot expand in powers of $\frac{g^2}{tc}$ so that we must do:

- a) Obtain exact consequences. Principally.
 - i) zero energy theorems (poles) (from perturbation theory)
 - ii) dispersion relations.
- b) Use approximations well defined and physical. Principally:
 - i) $\frac{M}{m} = 0.15$ It will be used as an expansion parameter
 - ii) exploit that area under resonance is large.

Outline of the course.

1. Phenomenology
 - a) kinematics (all conservation laws)
 - b) dynamics (interaction and cross sections)
2. Field Theory.
 - a) Perturbation theory.
 - b) dispersion relations.
 - c) applications.

Experimental facts.

We will consider: γ , N (u, p), π (π^\pm, π^0)

Masses:

$$\mu(\pi^\pm) = 273 \text{ me} \quad \mu(\pi^-) - \mu(\pi^0) \sim 9 \text{ me}$$

Lifetimes:

$$\pi^\pm \rightarrow \mu^\pm + \nu \quad 2 \sim 10^{-8} \text{ sec.}$$

But the characteristic lifetime is:

$$\tau_{ch} \sim \frac{t/c}{c} \sim \frac{10^{-13}}{10^{10}} = 10^{-23}$$

$\therefore \tau/\tau_{ch} \sim 10^{15}$ which is very large.

(in the atomic case $\tau/\tau_{ch} \sim 10^6$)

$$\pi^0 \rightarrow 2\gamma$$

$$5 \times 10^{-19} \leq \tau \leq 5 \times 10^{-16} \text{ sec.}$$

↑
from uncertainty principle.

Spin:

The angular momentum is conserved in nuclear physics. Now, if π is source of interaction, the conservation of nuclear ang. mom. can be accidental or the meson-nucleon system conserve its angular momentum. Therefore the meson must be assigned a quantum number for its ang. mom. at rest, i.e., its spin.

The spin of the π^\pm is determined by detailed balance considerations. Let the processes:

$$\rho_2 : \pi^\pm + D \rightarrow p + p$$

$$\rho_1 : p + p \rightarrow \pi^\pm + D$$

From the scattering theory (e.g. Lippmann-Schwinger or Goldmann-Goldberger papers) we have: for the initial process:

$$\Omega = \frac{2\pi}{VH^2} \sum_{\text{final}} |\mathcal{T}|^2 \rho_F(E)$$

By time reversal:

$$\mathcal{T}(\text{process}) = \mathcal{T}(\text{time-reversed process})$$

We have analogous expression for the final process. Then:

$$\frac{\rho_2}{\rho_1} = \frac{p^2}{\pi^2} \cdot \frac{4}{(2S+1)3}$$

where p and π are the momenta for p and π respectively in the CM system.
4 appears for the spin, S is the π spin of π^\pm and 3 for the spin of D .

By measurements: $S=0$

For the spin of π^0 we know that cannot be 1 because a neutral particle of spin 1 cannot decay in two photons.

The spin of π^0 seems to be also 0. This is consistent with the experimental data.

Parity:

The parity is conserved in Nuclear processes. We consider:
 $\pi^0 \rightarrow 2\gamma$.

For the E.M. field we can form two invariants:

$$E^2 - B^2 \quad \text{and} \quad E \cdot B$$

The 1st is scalar and the 2nd pseudoscalar. The 1st gives to parallel polarization and the 2nd to perpendicular polarization.

The parity is odd from experiments. But these experiments are actually realized with π^- mesons in the reaction:

$$\pi^- + p \rightarrow n$$

But we have also the reaction:

$$\pi^0 + p \rightarrow p$$

that represents the same effect in view of the charge independence.

The experiments are realized by π^- capture by deuterons forming magic atoms: $\pi^- + D \rightarrow$ Bohr orbit around D.

The first term has angular momentum $J=1$ and orbital parity (+) (in the lowest Bohr orbit).

We have also, in competition, the process of formation of 2 neutrons; In this case the angular momentum is $J=1$ and compatible with this only exist the triplet state (state P) given parity (-). Then the parity of the π^- is odd. (intrinsic parity).

Charge independence and isotopic spin.

We know also from the Nuclear physics the charge independence holds. We introduce the isotopic spin Ξ_3 . We define this as:

$$\Xi_3 = \begin{cases} +1 & \text{for protons (with eigenvalue } \alpha) \\ -1 & \text{for neutrons. (} \beta \text{)} \end{cases}$$

We use an extended Pauli principle which includes also the isotopic spin. The charge independence implies that the hamiltonian must be invariant in 2 space rotations. For the part 2 for we can construct the invariants:

$$\text{total } \Xi_3 = 1$$

The isotopic spin is: (for nuclear system)

$$I = \sum_i \Xi_3$$

and the covariance requirement is:

$$[I, H] = 0$$

If this is valid in presence of mesonic phenomena we ~~can~~ extend to the total system. I is the total isotopic spin of the system and must be conserved.

We consider the processes:

$$\begin{aligned} p &\rightarrow n + \pi \\ \frac{1}{2} &\rightarrow -\frac{1}{2} \\ p &\rightarrow p + \pi \\ \frac{1}{2} &\rightarrow \frac{1}{2} \\ n &\rightarrow p + \pi \\ -\frac{1}{2} &\rightarrow \frac{1}{2} \end{aligned}$$

from the $N \rightarrow N + \pi$ phenomena we must assign $I=1$ for charged π 's.

For the deuteron we have $T=0$. Then: D (state, triplet, singlet)
 Then for:

$$\pi + D \quad I=1$$

and in the experiments:

$$1st: \quad p + p \rightarrow \pi^+ + D$$

$$\text{we have } T=I=1 \\ I_3=1$$

$$2nd: \quad p + n \rightarrow \pi^0 + D \quad (I=1, I_3=0)$$

For the $p + n$ system we have:

$$\frac{1}{\sqrt{2}} \left(\frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \left(\frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\sqrt{2}} \right)$$

$$\text{triplet} \quad \text{singlet} \\ I=1 \quad T=1 \quad T=0, I=0$$

From this we can predict (only by the assumption of charge independence)

$$\frac{d\sigma(pp \rightarrow \pi^+ + D)}{d\sigma(pn \rightarrow \pi^0 + D)} = 2$$

and this is in agreement with experiments.

We assume that the charge independence is correct (at least at the point of our approach).

Antiparticles:

We consider as anti- π mesons:

$$\bar{\pi}^+ = \pi^+$$

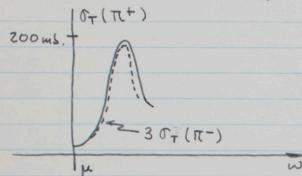
$$\bar{\pi}^0 = \pi^-$$

$$\bar{\pi}^- = \pi^0$$

~~possibly~~ this choice of antiparticles is not right. The main doubt comes from the π^0 's.

Interaction between pions and protons:

We consider as target a proton. The interaction takes place at resonance. In this: $J = \frac{3}{2}$, $I = \frac{3}{2}$, P state $\omega \approx 200$ Mev.



$$\text{Note: } 1 \text{ mb} = 10^{-27} \text{ cm}^2$$

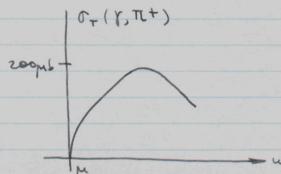
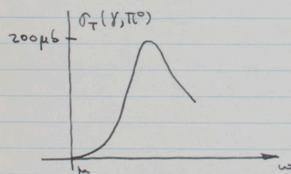
$$\left(\frac{\text{mb}}{\mu\text{c}}\right)^2 = 20 \text{ mb.}$$

$$\begin{aligned}\frac{\sigma(\pi^+)}{\sigma(\pi^- \text{ and } \pi^0)} &= \frac{9}{1} \\ \frac{\sigma(\pi^-)}{\sigma(\pi^0)} &= \frac{2}{1} \quad \text{at} \quad T_{\text{lab}} = \omega_L - \mu \gtrsim 50 \text{ Mev.} \\ &\qquad\qquad\qquad \lesssim 300 \text{ Mev.}\end{aligned}$$

The angular distribution is:

$$\frac{d\sigma}{d\Omega}(\pi^+ + p) \approx 1 + 3 \cos^2 \theta \quad (\text{P state } J = \frac{3}{2})$$

Photo production of pions.



This is an endothermic reaction with infinite slope at threshold ($T \approx \sqrt{\omega} - \mu$)

Problem 1: Work out the ratio:

$$\begin{aligned}\frac{d\sigma}{d\omega}(\pi^+ + D \rightarrow 2p) &= f(p, q, s) \\ \frac{d\sigma}{d\omega}(2p \rightarrow \pi^+ + D) &\end{aligned}$$

We want now describe the scattering of particles.

If both particles have spin zero we know as a very general result:

$$\frac{d\sigma}{d\omega} = |f|^2$$

If we use the partial waves expansion:

$$f = \sum_{l=0}^{\infty} (2l+1) f_l(q) P_l(\cos \theta)$$

we have for the elastic scattering:

$$f_l = \frac{e^{il\theta} \sin \delta_l}{q}$$

where q is the momentum in the C of Mass system and δ_l is real.
For the scattering of two spinless particles, the scattering amplitude must be a function of the only two invariants under rotations q^2 and $\vec{q}_f \cdot \vec{q}_i$.
 $\therefore f(q^2, \vec{q}_f \cdot \vec{q}_i) \quad (\vec{q}_f \cdot \vec{q}_i = q^2 \cos \theta)$

We will show the P_l (Legendre polynomials) are projection operators, i.e.:

$$\int \frac{d\omega}{4\pi} (2l+1) P_l(\vec{q}_f \cdot \vec{q}_i) (2l+1) P_l(\vec{q}_i \cdot \vec{q}_o) = (2l+1) P_l(\vec{q}_f \cdot \vec{q}_o)$$

where the integration is carried out for the intermediate states.

The proof is direct considering the addition theorem:

$$\begin{aligned}P_l(\vec{q}_i \cdot \vec{q}_o) &= \frac{4\pi}{2l+1} \sum_m Y_{lm}(\vec{q}_i \cdot \vec{q}_{i,d}) Y_{lm}^*(\vec{q}_o \cdot \vec{q}_{i,d}) \\ &= P_l(\vec{q}_i \cdot \vec{q}_f) P_l(\vec{q}_o \cdot \vec{q}_f) + \sum_{m \neq 0} Y_{lm} Y_{lm}^* \left(\frac{4\pi}{2l+1} \right)\end{aligned}$$

$$\text{and: } \int \frac{d\omega}{4\pi} P_l^2 = \frac{1}{2l+1}$$

Then we can expand in terms of projection operators.

For particles of spin $\neq 0$ we can construct a similar complete set of projection operators. We consider now a particle of spin zero and another one of spin $\frac{1}{2}$. The diff. cross section will be:

$$\frac{d\sigma}{d\omega} = \sum_{\alpha} |F_{\alpha}|^2 \beta$$

where α is the final and β the initial state.

If we require F must be invariant under rotations, etc. we will have:

$$F(\vec{q}_f \cdot \vec{q}_o, i \vec{p}_f \cdot \vec{q}_f \times \vec{q}_o)$$

(terms $\vec{p}_f \cdot \vec{q}_f$ and $\vec{p}_o \cdot \vec{q}_o$ are pseudoscalars and all quadratic term in \vec{p} can be reduced to a linear)

Then we can write the scattering amplitude as:

$$f_1(\vec{q}_f \cdot \vec{q}_o) + i \vec{p}_f \cdot \vec{q}_f \times \vec{q}_o f_2$$

This differs of the previous for the spin-orbit term.

We also want construct the projection operators in order to obtain a partial waves expansion. Since we have now two terms corresponding to $J = \frac{3}{2}$ and $J = \frac{1}{2}$, we take

$$1 = \frac{\vec{l} - \vec{P} \cdot \vec{l}}{2l+1} + \frac{\vec{l} + 1 + \vec{P} \cdot \vec{l}}{2l+1}$$

$$\therefore (2l+1) P_e = (\vec{l} - \vec{P} \cdot \vec{l}) P_e + (\vec{l} + 1 + \vec{P} \cdot \vec{l}) P_e \quad (a)$$

The way to remember this is:

$$\text{since: } \vec{j} = \frac{1}{2} \vec{l} + \vec{P}.$$

$$j(j+1) = \frac{3}{4} + l(l+1) + \vec{P} \cdot \vec{l} \quad \therefore \vec{P} \cdot \vec{l} = j(j+1) - l(l+1) - \frac{3}{4}$$

Therefore:

$$\text{For } j = l + \frac{1}{2} \quad \vec{P} \cdot \vec{l} = l$$

$$\text{and for } j = l - \frac{1}{2} \quad \vec{P} \cdot \vec{l} = -(l+1)$$

From (a) we see that the 1st term projects on $j = l - \frac{1}{2}$ and the 2nd on $j = l + \frac{1}{2}$. We prove now these are projection operators, i.e.:

$$\int (l - \vec{P} \cdot \vec{l}) P_e(\vec{q}_f, \vec{q}_i) (l - \vec{P} \cdot \vec{l}) P_e(\vec{q}_i, \vec{q}_o) \frac{d\Omega}{4\pi} = (l - \vec{P} \cdot \vec{l}) P_e(\vec{q}_f, \vec{q}_o)$$

Since $(l - \vec{P} \cdot \vec{l})$ is a selfadjoint operator we have:

$$\int (l - \vec{P} \cdot \vec{l})^2 P_e(\vec{q}_f, \vec{q}_i) (l - \vec{P} \cdot \vec{l})^2 P_e(\vec{q}_i, \vec{q}_o) \frac{d\Omega}{4\pi} = (l - \vec{P} \cdot \vec{l}) P_e(\vec{q}_f, \vec{q}_o)$$

using a similar calculation as previous. But:

$$(l - \vec{P} \cdot \vec{l})^2 = l^2 + (\vec{P} \cdot \vec{l})(\vec{P} \cdot \vec{l}) - 2l\vec{P} \cdot \vec{l} = l^2 + l(l+1) - 2l\vec{P} \cdot \vec{l} + i\vec{P} \cdot \vec{l} \times \vec{l} = l^2 + l(l+1) - 2l\vec{P} \cdot \vec{l} - \vec{P} \cdot \vec{l} = l(2l+1) - \vec{P} \cdot \vec{l}(2l+1)$$

using the formula: $(\vec{A} \cdot \vec{B})(\vec{C} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{A} \cdot \vec{B} \times \vec{C}$ (for any vectors \vec{A} and \vec{B} commuting or no) and $\vec{l} \times \vec{l} = i\vec{l}$

Then we have a complete and orthogonal set of projection operators. $\text{Kef. Since } \vec{l} = \vec{q}_f \times \frac{\nabla q_f}{i}$ (Fourier transform of the momentum)

$$\vec{P} \cdot \vec{l} P_e = \vec{P} \cdot \vec{q}_f \times \frac{\nabla q_f}{i} P_e = \vec{P} \cdot \vec{q}_f \times \frac{\nabla_{q_i}(\vec{q}_f, \vec{q}_o)}{i} P_e' = \frac{\vec{P} \cdot \vec{q}_f \times \vec{q}_o}{i} P_e'$$

We call: $l+ \equiv l + \frac{1}{2}$ $l- \equiv l - \frac{1}{2}$

We found the projection operators:

$$P_{e+} = (l+ + \vec{P} \cdot \vec{l}) P_e(\vec{q}_f, \vec{q}_o) = (l+ + 1) P_e - i\vec{P} \cdot \vec{q}_f \times \frac{\nabla q_o}{q_f q_o} P_e' = (l+ + 1) P_e - i(\vec{P} \cdot \hat{n}) \sin \theta P_e'$$

$$P_{e-} = (l- - \vec{P} \cdot \vec{l}) P_e(\vec{q}_f, \vec{q}_o) = l P_e + i\vec{P} \cdot \vec{q}_f \times \frac{\nabla q_o}{q_f q_o} P_e' = l P_e + i(\vec{P} \cdot \hat{n}) \sin \theta P_e'$$

calling $\hat{n} = \frac{\vec{q}_f \times \vec{q}_o}{|q_f \times q_o|}$. This (unit vector, normal to scattering plane)

The parity of the states $j = l \pm \frac{1}{2}$ is $(-)^l$. We mean by parity the orbital parity.

The scattering amplitude is:

$$F = f_1 + i\vec{P} \cdot \vec{q}_f \times \frac{\vec{q}_o}{q_f q_o} f_2 = \sum_l [f_{l+}(q) P_{e+} + f_{l-}(q) P_{e-}]$$

where $q = |\vec{q}_f| = |\vec{q}_o|$. The first values are:

j	l	Parity	P	State
$\frac{1}{2}$	0	+	$P_{0+} = 1$	$S_{1/2}$
$\frac{1}{2}$	1	-	$P_{1-} = \cos \theta + i(\vec{P} \cdot \hat{n}) \sin \theta$	$P_{1/2}$
$\frac{3}{2}$	1	-	$P_{1+} = 2 \cos \theta - i(\vec{P} \cdot \hat{n}) \sin \theta$	$P_{3/2}$
$\frac{3}{2}$	2	+	$P_{2-} = \dots$	$D_{3/2}$

The scattering amplitude for S, P states is:

$$F = [f_{S_{1/2}} + \cos \theta (f_{P_{1/2}} + 2f_{P_{3/2}})] + i\vec{P} \cdot \hat{n} \sin \theta (f_{P_{1/2}} - f_{P_{3/2}})$$

The cross-section is:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\alpha \beta} |F_{\alpha \beta}|^2$$

$$\text{But: } \frac{1}{2} \sum_{\alpha \beta} F_{\alpha \beta} F_{\alpha \beta}^* = \frac{1}{2} \sum_{\alpha \beta} F_{\alpha \beta} (F^+)_{\alpha \beta} = \frac{1}{2} \sum_{\alpha} (FF^+)_\alpha = \frac{1}{2} \text{Tr}(FF^+)$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{1}{2} \text{Tr}(FF^+)$$

$$\text{Since } F = f_1 + i\vec{P} \cdot \hat{n} \sin \theta f'_2 \quad F^+ = f'_1 - i\vec{P} \cdot \hat{n} \sin \theta f'_2$$

$$\text{and: } \text{Tr } \vec{P} = 0, \quad (\vec{P} \cdot \vec{n})(\vec{P} \cdot \vec{n}) = 1$$

$$\text{we have: } \frac{1}{2} \text{Tr}(FF^+) = \frac{1}{2} \text{Tr}(|f_1|^2 + \sin^2 \theta |f'_2|^2) = |f_1|^2 + |f'_2|^2 \sin^2 \theta$$

$$\therefore \frac{d\sigma}{d\Omega} = |f_{S_{1/2}} + \cos \theta (f_{P_{1/2}} + 2f_{P_{3/2}})|^2 + \sin^2 \theta |f_{P_{1/2}} - f_{P_{3/2}}|^2$$

We have a flat angular distribution for $S_{1/2}$ and $P_{1/2}$ states. For $P_{3/2}$ we have:

$$\left(\frac{d\sigma}{d\Omega} \right)_{P_{3/2}} \sim 4 \cos^2 \theta + \sin^2 \theta = 1 + 3 \cos^2 \theta$$

At low energies:

$$f_{l+}(q) P_{e+} \left(\frac{\vec{q}_f \cdot \vec{q}_o}{q_f q_o} \right) \rightarrow \left(\frac{\vec{q}_f \cdot \vec{q}_o}{q_f^2} \right)^l + \text{lower terms.}$$

But we expect finite results in the limit $q \rightarrow 0$ for forces of finite range (For example for Yukawa forces we have terms as:

$\frac{1}{(\vec{q}_f - \vec{q}_o)^2 + \mu^2}$ (Fourier transform) and it is finite for $\vec{q}_f \rightarrow \vec{q}_o$)

Then we have, as an effect of the angular momentum barrier, that for small \vec{q} : $f_e(q) \sim q^{2\ell}$

Problem 2. Calculate the polarization of recoil proton in $\pi + p$ scattering for S and P waves only.

Note: the polarization is:

$$p = \frac{1}{2} \sum_{\text{polar}} F_{\text{polar}}^+ (\vec{q}) \frac{d\vec{\sigma}}{d\Omega}$$

Please shift:

The S matrix is: (Cf. Gellmann-Goldberger or Lippmann-Schwinger papers)

$$S_{px} = 1_{px} - i(2\pi)^4 \delta(\Delta E) \delta(\Delta \vec{p}) T_{px}$$

and the cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{V_1 + V_2} \int \frac{1}{2} (2\pi)^4 \delta(\Delta E) \delta(\Delta \vec{p}) |T_{px}|^2$$

where 2 stands specifying the particular event in what we are interested (e.g. scattering, absorption etc.), and:

$$2\pi \delta(\Delta E) = \lim_{T \rightarrow \infty} \int_{-T}^T e^{i\Delta Et} dt$$

That is S gives the state at time T if we know it at time -T for T large. Since we require conservation of probability:

$$(S_{+}, S_{+}) = (t_0, t_0)$$

$$\therefore S^+ S = 1 \quad (\text{nucleon-nucleon scattering})$$

We apply this to two particles A with mass μ and B with mass m in a bath whose collision is described in the C of M system.

The number of states is:

$$2\pi (2\pi)^3 \int \int \frac{d\vec{p}_f d\vec{q}_f}{(2\pi)^3 (2\pi)^3} \delta(\vec{p}_f + \vec{q}_f) \delta(E_f + \omega_f - E_o - \omega_o) =$$

$$= 2\pi \int \frac{d\vec{q}_f}{(2\pi)^3} \delta(E_f + \omega_f - E_o - \omega_o)$$

$$\text{But: } E_f = \sqrt{\mu^2 + \vec{q}_f^2} \quad \omega_f = \sqrt{\mu^2 + \vec{q}_f^2}$$

Then the differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{V_1 + V_2} \int \frac{\vec{q}_f^2 d\vec{q}_f}{(2\pi)^3} \frac{1}{dW} \delta(\underbrace{\sqrt{\mu^2 + \vec{q}_f^2} + \sqrt{\mu^2 + \vec{q}_f^2} - E_o - \omega_o}_{W} - E_o - \omega_o) dW |T|^2$$

↳ Here angle and energy are independent (only in the C.M. system and not true in laboratory system e.g. Compton effect)

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\frac{q_0}{\omega_0} + \frac{q_0}{E_0}} \frac{q_0^2}{(2\pi)^3} \left. \frac{1}{dW} \right|_{q_f = q_0} |T|^2 = \frac{1}{(\frac{1}{\omega_0} + \frac{1}{E_0})^2} \frac{1}{(2\pi)^2} |T|^2$$

$$\therefore \frac{d\sigma}{d\Omega} = \left| \frac{T}{2\pi} \frac{\omega E_0}{\omega + E_0} \right|^2$$

Then we have the identification:

$$T = -\frac{2\pi(\omega + E_0)}{\omega E_0} F$$

where the sign is chosen as - by standard convention.

We apply now the unitary condition:

$$S^+ S = 1$$

$$[1 + i(2\pi)^4 \delta^4(-T)] [1 - i(2\pi)^4 \delta^4(-T)] = 1$$

$$\therefore (2\pi)^4 \delta(-T + (2\pi)^4 \delta(-T) + i(2\pi)^4 \delta(-T)) (T + T) = 0$$

↑ provided we keep the energy const.

Then in terms of the scattering amplitude F:

$$\frac{2\pi(\omega + E)}{\omega E} F^+ (2\pi)^4 \delta(-T) F + (-i)(F^+ - F) = 0$$

$$\frac{2\pi(\omega + E)}{\omega E} F^+ (2\pi)^4 \delta[\omega(q_f) + E(q_f) - \omega - E] \frac{d\vec{q}_f}{(2\pi)^3} F - i(F^+ - F) = 0$$

$$\frac{1}{2\pi} \frac{\omega + E}{\omega E} \frac{\vec{q}^2}{(2\pi)^3} \int d\Omega i F^+ F - i(F^+ - F) = 0$$

Thus we have:

$$\frac{1}{2\pi} \int d\Omega i F^+ F - i(F^+ - F) = 0$$

where we have allowed no non-elastic intermediate states.

Reading: R.P. Feynman Phys Rev. 1949: Theory of Particles and Space Time approach to Quantum Mechanics.

We have found: (for no inelastic process).

$$\frac{F - F^+}{i} = \int d\Omega F^+ F \frac{q}{2\pi}$$

But we have expanded the scattering amplitude in partial waves:

$$F = \sum_{e,j} P_{e,j} f_{e,j}$$

where:

$$P_{e,j}^+ = P_{e,j} \quad \text{and} \quad \int \frac{d\Omega}{4\pi} P^2 = P$$

Then, our unitarity condition becomes:

$$\sum_{e,j} P_{e,j} \frac{f_{e,j}(q) - f_{e,j}^*(q)}{2i} = q \int \frac{d\Omega}{4\pi} \sum_{e,j} P_{e,j} f_{e,j}^* \sum_{e',j'} P_{e',j'} f_{e',j'}(q) = q \sum_{e,j} P_{e,j} f_{e,j}^* f_{e,j}$$

Thus we have:

$$\left| \operatorname{Im} f_{e,j} \right| = q |f_{e,j}|^2$$

The unique solution of this equation is:

$$f_{e,j} = \frac{e^{i\delta_{e,j}} \sin \delta_{e,j}}{q}$$

with $\delta_{e,j}$ real. Now the total cross section is:

$$\sigma_T = \frac{1}{2} \sum_{\alpha} \int \langle \alpha | F^+(q_f + q_0) F(q_f, q_0) | \alpha \rangle d\Omega_f =$$

$$= 4\pi \sum_{e,j} |f_{e,j}|^2 \frac{1}{2} \sum_{\alpha} \langle \alpha | P_{e,j}(q_f, q_0) | \alpha \rangle$$

But we recall

$$P_{e+} = (l+1) P_e - \frac{i \vec{\sigma} \cdot \vec{q}_f \times \vec{q}_0}{\vec{q}_f \cdot \vec{q}_0} P'_e$$

$$P_{e-} = l P_e + \frac{i \vec{\sigma} \cdot \vec{q}_f \times \vec{q}_0}{\vec{q}_f \cdot \vec{q}_0} P'_e \quad \text{vanish to } \vec{q}_f = \vec{q}_0$$

and $P_e(1) = 1$. Thus we have:

$$\sigma_T = 4\pi \sum_{e,j} |f_{e,j}|^2 (j + \frac{1}{2}) =$$

$$= 4\pi \lambda^2 \sum_{e,j} (j + \frac{1}{2}) \sin^2 \delta_{e,j} \quad \text{for the elastic case.}$$

For the case of non-elastic scattering we would obtain:

$$\operatorname{Im} f_{e,j} = q \left[|f_{e,j}|^2 + \frac{P_{e,j}}{4\pi(l + \frac{1}{2})} \right]$$

One can obtain inequalities from this relation:

$$f = a + ib \quad c = \operatorname{Im} f$$

$$\therefore b = a^2 + b^2 + c^2 \quad \text{and} \quad a^2 + (b - \frac{1}{2})^2 + c^2 = \frac{1}{4}$$

We must also include the effect of isotopic spin by analogy to the P state.

We can already describe

$$\pi^+ + p \rightarrow \pi^+ + p$$

because there are no other intermediate states.

$$\text{But for: } \pi^- + p \rightarrow \pi^- + p \rightarrow \pi^0 + n$$

So far we cannot account for the inelastic π^0 scattering. Here also we have a conservation law for total isotopic spin.

For angular momentum:

$$l = 1 \quad x, y, z$$

$$\begin{array}{ll} l_z = +1 & \frac{x+iy}{\sqrt{2}} \\ l_z = 0 & z \\ l_z = -1 & \frac{x-iy}{\sqrt{2}} \end{array} \Rightarrow \begin{array}{ll} I_z = 1 & Q = +1 \\ I_z = 0 & Q = 0 \\ I_z = -1 & Q = -1 \end{array}$$

Thus the pion wave functions will be:

$$\Psi_{\alpha} : \alpha = 1, 2, 3$$

$$\Psi_{(+)} \equiv \frac{\Psi_1 + i\Psi_2}{\sqrt{2}}$$

$$\Psi_{(-)} \equiv \frac{\Psi_1 - i\Psi_2}{\sqrt{2}}$$

To obtain the projection operator we consider the angles:

$$P_{\frac{1}{2}} : \vec{q}_f \cdot \vec{q}_0 + i \vec{\sigma} \cdot \vec{q}_f \times \vec{q}_0 = \vec{\sigma} \cdot \vec{q}_f \vec{\sigma} \cdot \vec{q}_0$$

$$\text{For } I = \frac{1}{2} \quad \vec{\sigma} \rightarrow \vec{z} \quad \vec{q} \rightarrow \alpha$$

$$\langle \beta | I_{\frac{1}{2}} | \alpha \rangle = \frac{2\beta Z \alpha}{3}$$

where 3 is the normalization to make projection operator cancel out in unitarity relations.

For $3/2$ state:

$$P_{3/2} : 2\vec{q}_f \cdot \vec{q}_0 - i \vec{\sigma} \cdot \vec{q}_f \times \vec{q}_0 = 3\vec{q}_f \cdot \vec{q}_0 - \vec{\sigma} \cdot \vec{q}_f \vec{\sigma} \cdot \vec{q}_0$$

$$I = \frac{3}{2} \quad \langle \beta | I_{3/2} | \alpha \rangle = 5\beta \alpha - \frac{1}{3} Z \beta Z \alpha$$

We now verify the normalization by computing:

$$\langle \gamma I_{1/2}^+ I_{1/2}^- | \alpha \rangle = \sum_{\beta} \frac{z_\gamma z_\beta}{3} \frac{z_\beta z_\alpha}{3} = \frac{1}{3} z_\gamma z_\alpha = \langle \gamma I_{1/2}^- | \alpha \rangle$$

We must now compute matrix elements for various processes:

$$I = \frac{1}{2} : \quad \frac{1}{3} z_\beta z_\alpha$$

$$\pi^+ p : \quad \frac{1}{3} z_\beta z^+ | p \rangle = 0$$

$$\pi^- p : \quad \frac{\sqrt{2}}{3} z_\beta z^- | p \rangle = \frac{\sqrt{2}}{3} z_\beta | u \rangle$$

$$\text{where: } z^+ \equiv \frac{z_1 + i z_2}{\sqrt{2}} \quad z^- \equiv \frac{z_1 - i z_2}{\sqrt{2}}$$

Thus we have:

$$\langle I_{1/2}^- | \pi^+ p \rangle = 0$$

$$\langle \pi^- p | I_{1/2}^- | \pi^+ p \rangle = \frac{\sqrt{2}}{3} \langle p | \frac{z_1 + i z_2}{\sqrt{2}} | u \rangle \cdot \frac{\sqrt{2}}{3} = \frac{2}{3}$$

$$\langle \pi^+ u | I_{1/2}^- | \pi^- p \rangle = \frac{\sqrt{2}}{3} \langle u | z_3 | u \rangle = -\frac{\sqrt{2}}{3}$$

For the $I = \frac{3}{2}$ state we have:

$$(I = \frac{3}{2}) = 1 - (I = \frac{1}{2})$$

so that:

$$\langle \pi^+ p | I_{3/2}^- | \pi^+ p \rangle = 1$$

$$\langle \pi^- p | I_{3/2}^- | \pi^- p \rangle = \frac{1}{3}$$

$$\langle \pi^+ u | I_{3/2}^- | \pi^- p \rangle = \frac{\sqrt{2}}{3}$$

We note that for $I = \frac{3}{2}$ only:

$$\frac{\langle \pi^+ p |}{\langle \pi^- p |} = \frac{1}{\frac{3}{4}} = \frac{4}{3} \quad \text{which is observed experimentally.}$$

Now we must write the total scattering amplitude:

$$F = I_{3/2} \sum_{l,j} f_{l,j;3/2}(\vec{q}) P_{lj} + I_{1/2} \sum_{l,j} f_{l,j;1/2}(\vec{q}) P_{lj}$$

We use the notation

$$\delta_{2z,2j}$$

Experimentally:

$$S_{33} = \text{resonance} \rightarrow \frac{\pi}{2} \text{ at 200 MeV.}$$

$$S_3 = -0.11 \text{ g/}\mu$$

$$S_1 = 0.16 \text{ g/}\mu$$

The other P waves S_{31}, S_{13}, S_{11} are small and can be set equal to zero at 3% which is within limits of charge independence.

Field Theory.

There are two approaches:

1) Canonical approach: consider a field as a quantum mechanical operator with infinite number of degrees of freedom.

2) Introduce a field by definition. We'll follow this approach.

We now consider a system with

1) Neutral scalar mesons (mass μ)

2) Bose statistics.

For non-interacting particles we can write down the states of the system.

$$\Phi_0, \Phi_{\vec{k}}, \Phi_{\vec{k}_1, \vec{k}_2, \dots} \quad (\text{two particle state})$$

(vacuum) \downarrow (one particle state)

We normalize state. The states have the properties:

$$(\Phi_0, \Phi_0) = 1 \quad (\text{normalization})$$

$$(\Phi_{\vec{k}}, \Phi_{\vec{k}'}) = (2\pi)^3 \delta(\vec{k} - \vec{k}') \quad (\text{corresponds to choosing } 1 \cdot e^{i\vec{k} \cdot \vec{x}})$$

Now we claim:

$$1 = \Phi_0 \langle \Phi_0 | + \int \frac{d\vec{k}}{(2\pi)^3} \Phi_{\vec{k}} \langle \Phi_{\vec{k}} | + 2p + \dots$$

Proof:

$$1 \cdot \Phi_0 = \Phi_0$$

$$1 \cdot \Phi_{\vec{k}} = \int \frac{d\vec{k}'}{(2\pi)^3} \Phi_{\vec{k}'} (\Phi_{\vec{k}}, \Phi_{\vec{k}'}) = \int \frac{d\vec{k}'}{(2\pi)^3} (2\pi)^3 \delta(\vec{k} - \vec{k}') \Phi_{\vec{k}'} = \Phi_{\vec{k}}$$

We can now define the creation and destruction operators (This has nothing to do with field theory)

Let $a_{\vec{k}}^*$ create a particle of momentum \vec{k}

$$\Phi_{\vec{k}} = a_{\vec{k}}^* \Phi_0 \quad (\text{this defines the normalization of } a_{\vec{k}} \text{ but not the phase})$$

$$\Phi_0 = a_{\vec{k}} \Phi_0$$

$$a_{\vec{k}} \Phi_0 = 0$$

We can obtain the commutation rules of a_k from:

$$(2\pi)^3 \delta(\vec{k}' - \vec{k}) = (\Phi_{k'}, \Phi_k) = (\Phi_0, a_k a_k^* \Phi_0) \\ = (\Phi_0, [a_k, a_k^*] \Phi_0)$$

Then we can write:

$$[a_k, a_k^*] = (2\pi)^3 \delta(\vec{k} - \vec{k})$$

We now want to define the number operator. Let:

$$n_k^2 = a_k^* a_k$$

Now we claim: $n(\Delta) = \int \frac{d\vec{k}}{(2\pi)^3} = \text{number of particles with momentum } \vec{k} \text{ in } \Delta$

Proof: $[n(\Delta), a_k] = \begin{cases} -a_k & \text{if } k \text{ is in } \Delta \\ 0 & \text{if } k \text{ not in } \Delta \end{cases} \quad (t=c=1)$

Thus:

$$a_k \Phi_n = \Phi_{n-1}$$

Now we can define the energy and momentum

$$E = \int \frac{d\vec{k}}{(2\pi)^3} n(\vec{k}) \omega(k) = H$$

$$\vec{P} = \int \frac{d\vec{k}}{(2\pi)^3} n(\vec{k}) \vec{k} \quad \omega = \sqrt{\mu^2 + k^2}$$

Then (eqs. of motion) $\dot{a}_k = i[H, a_k] = -i\omega_k a_k$

$$a_k(t) = e^{-i\omega_k t} a_k(0)$$

$$n_k(t) = n_k(0)$$

Up to here there are no field theory. It is only a way of describe relativistically a system of particles.

We can now define a field using the notation

$$x = (\vec{x}, x_0) \quad d\vec{x} = 3 \text{ dim.}$$

$$dx = 4 \text{ dim.}$$

$$x \cdot y = \vec{x} \cdot \vec{y} - x_0 y_0$$

$$k = (\vec{k}, k_0)$$

$$\phi(x) \equiv \int \frac{d\vec{k}}{(2\pi)^3} \left[e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{a_k}{\sqrt{2\omega_k}} f(\vec{k}) + e^{-i(\vec{k} \cdot \vec{x} - \omega t)} \frac{a_k^*}{\sqrt{2\omega_k}} f^*(\vec{k}) \right]$$

We choose ϕ to be hermitian since we want a scalar boson field.

The importance of the field is to be able to write down an invariant interaction.

We must now discuss Lorentz invariance. (This must be required on the quantities)

One observer measures $\langle a | \phi(x) | b \rangle$

$$\langle a | b \rangle$$

Another observer measures $\langle a' | \phi(x') | b' \rangle$

because he has same theory.

These must be equal; now if we can write

$$b' = U b \quad a' = U a$$

$$\langle a' | \phi(x') | b' \rangle = \langle a | U^{-1} \phi(x') U | b \rangle$$

$$\therefore \phi(x) = U^{-1} \phi(x') U$$

Now if we have an invariant theory all c-numbers must be invariants. We can construct the commutator $[\phi(x), \phi(x')]$ which must be a c-number! (The right)

This will determine $f(\vec{k})$

Now, since we must have invariance under translations we must have

$$[\phi(x), \phi(y)] = \Delta(x-y)$$

Explicitly we have:

$$[\phi(x), \phi(y)] = \int \frac{d\vec{k}}{(2\pi)^3} \frac{|f(\vec{k})|^2}{2\omega_k} \left[e^{i[\vec{k} \cdot (\vec{x}-\vec{y}) - \omega_k(x_0-y_0)]} - e^{i[\vec{k} \cdot (\vec{x}-\vec{y}) + \omega_k(x_0-y_0)]} \right]$$

Since this is spherically symmetric we can change sign of \vec{k} in the second integral.

$$= \int \frac{d\vec{k}}{(2\pi)^3} \frac{|f(\vec{k})|^2}{2\omega_k} e^{i\vec{k} \cdot (\vec{x}-\vec{y})} \left[e^{-i\omega_k(x_0-y_0)} - e^{i\omega_k(x_0-y_0)} \right]$$

Now we note that:

$$\int \frac{d\vec{k}}{2\omega_k} = \int d\vec{k} dk_0 \delta(k^2 - \omega_k^2) \in(k_0) \quad \hookrightarrow \delta[(k_0 - \omega_k)(k_0 + \omega_k)] = \frac{\delta(k_0 - \omega_k)}{2\omega_k} - \frac{\delta(k_0 + \omega_k)}{2\omega_k}$$

Then:

$$\therefore [\phi(x), \phi(y)] = \int \frac{d\vec{k} dk_0}{(2\pi)^3} |f(\vec{k})|^2 \in(k_0) \delta(k^2 + \mu^2) e^{i\vec{k} \cdot (\vec{x}-\vec{y})}$$

For this to be a Lorentz invariant $|f(\vec{k})|^2 \in(k_0) \delta(k^2 + \mu^2)$ must be Lorentz invariant (proof omitted)

For $|k_0| > k$ the $\in(k_0)$ is invariant.

But $\delta(k^2 + \mu^2)$ makes k timelike so $\in(k_0) \delta(k^2 + \mu^2)$ is invariant.

$\therefore f(\vec{k})$ must be invariant and hence must be a constant. Thus we choose $f(\vec{k}) = 1$ (note: by arbitrary normalization, and then:

$$\phi(x) = \int \frac{d\vec{k}}{(2\pi)^3} \left[e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{a_k}{\sqrt{2\omega_k}} + e^{-i(\vec{k} \cdot \vec{x} - \omega t)} \frac{a_k^*}{\sqrt{2\omega_k}} \right]$$

(unique definition of the free-field)

We claim that for free particles there must exist a field but this tells us nothing new unless we have interacting particles.

We can now develop usual field theory with the definition and results.

$$\phi(x, t) \equiv \Pi(x, t)$$

$$\text{We can write: } [\pi(\vec{x}, t), \phi(\vec{x}', t)] = \frac{1}{i} \delta(\vec{x} - \vec{x}')$$

$$\text{and: } H = \int \left[\frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} [\nabla \phi(\vec{x})]^2 + \frac{1}{2} \mu^2 \phi^2(\vec{x}) \right] + \text{constant.}$$

Then we can construct

$$\mathcal{L} = \frac{1}{i} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - (\nabla \phi)^2 \right] - \frac{\mu^2}{2} \phi^2$$

and obtain the equations of motion:

$$(\square^2 - \mu^2) \phi(x) = 0 \quad \text{because } (\square^2 - \mu^2) e^{\pm i(\vec{k} \cdot \vec{x} - \omega t)} = 0$$

$$\text{where } \square^2 = \nabla^2 - \frac{\partial^2}{\partial t^2}$$

All of this follows from the existence of particles, Lorentz invariance and E-p relation.

Now in our case:

$$[\phi(x), \phi(y)] = 0 \quad \text{for spacelike separation.}$$

If $\phi(x)$ is a true observable we must have this since two measurements shouldn't interfere for space like separations.

We can prove the above since:

$$[\phi(x), \phi(y)] = \Delta(x-y) \text{ invariant.}$$

Now $\Delta(\vec{x}-\vec{y}, 0) = 0$ and since it is an invariant it must be true for all space-like separations.

We have:

$$\Delta(x-y) = [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k}(\vec{x}-\vec{y})} \frac{(e^{-i\omega_k(x_0-y_0)} - e^{i\omega_k(x_0-y_0)})}{2\omega_k}$$

We know that this is:

(a) Lorentz invariant.

b) $\Delta(x-y) = 0$ at $x_0 - y_0 = 0$ thus $\Delta(x) = 0$ for $(x-y)^2 < 0$ (spacelike)

$$c) \left[\frac{\partial \phi(\vec{x}, 0)}{\partial t}, \phi(\vec{y}, 0) \right] = \frac{1}{i} \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}(\vec{x}-\vec{y})} = \frac{1}{i} \delta(\vec{x}-\vec{y})$$

$\hookrightarrow = \pi(\vec{x}, 0)$

The plane wave decomposition is something inconvenient. It is easier to use wave packets which have physical significance.

We consider a set of functions $f_n(x)$ which satisfy the Klein-Gordon equation with positive frequency.

$$(\square^2 - \mu^2) f_n(x) = 0 \quad \text{complete set at one time} \rightarrow e^{-i\omega t}$$

We can construct a complete set of wave packets at one time at x_n

with momentum p_n .

We normalize the functions $f_n(x)$ by:

$$(f_n, f_m) = i \int [f_n^* \frac{\partial f_m}{\partial t} - \frac{\partial f_n^*}{\partial t} f_m] dx$$

This is sensible since we claim that it is a constant of time.

$$\begin{aligned} \frac{\partial}{\partial t} (f_n, f_m) &= i \int [f_n^* \frac{\partial^2 f_m}{\partial t^2} - \frac{\partial^2 f_n^*}{\partial t^2} f_m] dx \\ &= i \int [f_n^* \nabla^2 f_m - (\nabla^2 f_n)^* f_m] dx \quad (\text{since } \frac{\partial^2}{\partial t^2} = (\nabla^2 \mu^2) f) \\ &= i \int \nabla \cdot [f_n^* \nabla f_m - \nabla f_n^* f_m] dx = 0 \end{aligned}$$

if f is spatially limited.

$$\text{We choose } (f_n, f_m) = \delta_{nm}$$

Now we can construct creation operators in terms of these states.

$$a_n^+ = i \int d\vec{x} [\phi(x) \frac{\partial f_n}{\partial t} - \frac{\partial \phi}{\partial t} f_n] \rightarrow \text{positive freq.}$$

$$a_m^- = i \int d\vec{x} [f_m^* \frac{\partial \phi}{\partial t} - \frac{\partial f_m^*}{\partial t} \phi] \rightarrow \text{negative freq.}$$

We now calculate the commutator

$$\begin{aligned} [a_m^-, a_n^+] &= - \left[d\vec{x} d\vec{y} \left[f_m^* \frac{1}{i} \delta(\vec{x}-\vec{y}) \frac{\partial f_n}{\partial t} - \frac{1}{i} \delta(\vec{x}-\vec{y}) \frac{\partial f_n^*}{\partial t} f_m \right] \right] = \\ &= i \int d\vec{x} [f_m^* \frac{\partial f_n}{\partial t} - \frac{\partial f_m^*}{\partial t} f_n] = (f_n, f_m) = \delta_{nm} \end{aligned}$$

The point of this is that we now have normalized states.

Similarly:

$$[a_m^+, a_n^+] = \pm (f_n^*, f_m) = 0$$

$$[a_m^-, a_n^-] = \pm (f_n, f_m^*) = 0$$

since both are positive frequency and we set a positive freq. function which is a constant and hence zero.

Problem 3: Show: $(f_n, f_n) > 0$

Now we have in the back of our minds the limiting case:

$$f_n \sim \frac{e^{i k_n \vec{x}}}{\sqrt{2\omega}} \frac{e^{-i\omega t}}{V}$$

In the general case V is a function of \vec{x} and gives a truly normalizable state but not an eigenfunction of momentum

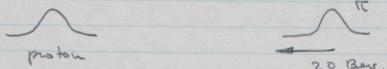
$$\phi(x) = \frac{1}{i} \sum_n (f_n a_n + f_n^* a_n^*) \quad \dot{\phi}(x) = \frac{1}{i} \sum_n \left(\frac{\partial f_n}{\partial t} a_n + \frac{\partial f_n^*}{\partial t} a_n^* \right)$$

Interacting fields.

It is convenient to use the Heisenberg representation. Here the physical state vectors are constant in time and the time dependence is

all in the operators.

It is convenient to take the simplest possible description of the state.

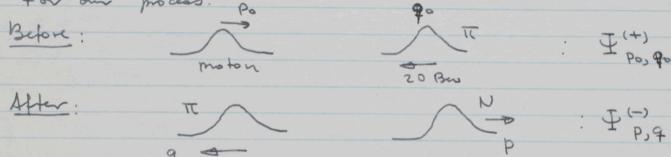


Simplest way is to describe them at a time when the state is simple.

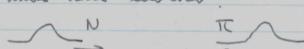
- 1) $\mathbb{I}_{[in]}^{(+)}$ corresponds to plane outgoing waves.
These are simple before collision.
- 2) $\mathbb{I}_{[in]}^{(-)}$ corresponds to plane incoming waves.
These are simple after collision.

The set $\{\mathbb{I}\}$ means a simple complete set of states. Simple means that we have one state in Fock space.

For our process.



What would we have had earlier?



But this would have scattering but we have wave so that there must be plane wave and incoming waves just before the collision to cancel out expected scattered state.

We now list the states:

- 1) Vacuum \mathbb{I}_0 (same for $(-)$ and $(+)$)
- 2) 1 particle states (same for $(-)$ and $(+)$)
- 3) 2 particle states:
 - a) $(-)$ and $(+)$ can be the same if the particles miss each other.
 - b) They can hit and hence $(+) \neq (-)$

We now consider the operators which can create and annihilate these states. The operators which do this are defined.

$$\mathbb{I}_{[in]}^{(+)} : a_n^{(in)}, a_n^{(in)} \rightarrow \phi^{(in)}(x) \text{ for collision at } \bar{x}=0, t=0$$

$$\mathbb{I}_{[in]}^{(-)} : a_n^{(out)}, a_n^{(out)} \rightarrow \phi^{(out)}(x) \text{ here are complete in the future.}$$

These operators satisfy free field commutation relations.

One can not really define $\phi^{(in)}(x)$, $\phi^{(out)}(x)$ for all t because at any time $\mathbb{I}^{(+)}, \mathbb{I}^{(-)}$ states are not really separable.

We can define the S matrix by:

$$\langle f | S | i \rangle = (\mathbb{I}_f^{(-)}, \mathbb{I}_i^{(+)})$$

One can forget about the wave packet and consider only the central momentum of wave packet if the cross section does not vary over the spread of the wave packet.

Now LSZ define or introduce field theory by assuming that there exist a field operator $\phi(x)$ which is invariant and which has the limits:

$$\phi_{\text{future}}^{\text{out}} \leftarrow \phi(x) \rightarrow \phi_{\text{past}}^{\text{in}}$$

We must be carefull and must use wave packet definition and thus not do the above.

We must use wave packet limits of the above field and must consider matrix elements between wave packet states.

$$\langle b | a_n^{\text{out}}(t) | a \rangle \xleftarrow[t \rightarrow +\infty]{\substack{\downarrow \\ \text{wave packet states}}} \langle b | a_n^{\text{in}}(t) | a \rangle \xrightarrow[t \rightarrow -\infty]{\substack{\downarrow \\ \text{at } t \rightarrow -\infty}}$$

The point is that wave packet states can be separated if for \mathbb{I}^{in} we go to sufficient early times.

We now consider the LSZ Reduction Theorem.

We must use the identity

$$i \int_{t_0}^{t_1} [(\Box^2 - \mu^2), \phi(x)] f_n(x) dx = ?$$

$$\downarrow$$

$$f_n(\Box^2 - \mu^2) \phi(x) - [(\Box^2 - \mu^2) f_n] \phi(x)$$

The space part vanishes by partial integration and using:

$$- f_n \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 f_n}{\partial t^2} \phi = \frac{\partial}{\partial t} \left[\frac{\partial f_n}{\partial t} \phi - f_n \frac{\partial \phi}{\partial t} \right]$$

$$\text{Thus: } i \int_{t_0}^{t_1} (\Box^2 - \mu^2) \phi(x) f_n(x) dx = \int_{t_0}^{t_1} dt \frac{\partial}{\partial t} i \int d\bar{x} \left[\frac{\partial f_n}{\partial t} \phi - f_n \frac{\partial \phi}{\partial t} \right]$$

But we defined:

$$a_n^* = i \int \left[\frac{\partial f_n(x)}{\partial t} \phi(x) - \frac{\partial \phi(x)}{\partial t} f_n(x) \right] d\bar{x}$$

$$\text{So: } i \int_{t_0}^{t_1} (\Box^2 - \mu^2) \phi(x) f_n(x) dx = \int_{t_0}^{t_1} dt \frac{\partial}{\partial t} a_n^*(t)$$

Now if we let $t_0 \rightarrow$ sufficient earlier such $a_n(t) \rightarrow a_n^{in}(t)$ and $t_1 \rightarrow$ suff. later such $a_n(t) \rightarrow a_n^{out}(t)$, then:

$$[a_{\text{in}}^{+ \text{out}} - a_{\text{in}}^{+ \text{in}}] = i \int_0^x (\square^2 - \mu^2) \phi(x) f_{qf}(x) dx$$

We will now consider boson-fermion scattering and will reduce this to a fermion-state, boson-operator form.

Then:

$$\langle f | S | i \rangle = \langle \Phi_f^{(-)}, \Phi_i^{(+)} \rangle = \langle \Phi_f^{(-)}, a_{qf}^{+ \text{in}} \Phi_{p0} \rangle$$

↑
in index
I-spin ↓ can drop (+) to single
particle state since (+)=(-)

Using our identity:

$$\langle f | S | i \rangle = \langle \Phi_f^{(-)}, [a_{qf}^{+ \text{out}} - i \int (\square^2 - \mu^2) \phi(x) f_{qf}(x) dx] \Phi_{p0} \rangle$$

But in our problem we know that:

$$\langle \Phi_f^{(-)} | = \langle \Phi_{p0} | a_{p0}^{+ \text{out}}$$

if we restrict ourselves to elastic scattering. Then:

$$\langle f | S | i \rangle = \langle a_{p0}^{+ \text{out}} a_{qf}^{+ \text{out}} \Phi_{p0}, \Phi_{p0} \rangle - \langle \Phi_{p0}, a_{qf}^{+ \text{out}} i \int (\square^2 - \mu^2) \phi(x) f_{qf}(x) dx \Phi_{p0} \rangle$$

$$\text{But } [a_{p0}^{+ \text{out}}, a_{qf}^{+ \text{out}}] = \delta_{p0, qf}$$

and $a_{qf}^{+ \text{out}} \Phi_{p0} = 0$. Thus we have:

$$\langle f | S | i \rangle = \delta_{p0, qf} \delta_{p0, p0} - \langle \Phi_{p0}, a_{qf}^{+ \text{out}} i \int (\square^2 - \mu^2) \phi(x) f_{qf}(x) dx \Phi_{p0} \rangle$$

Now we recall that in usual field theory:

$$(\square^2 - \mu^2) \phi = i g \bar{\psi}_5 + = J(x)$$

Thus here we define:

$$(\square^2 - \mu^2) \phi(x) = J(x)$$

Then not worrying about interchanging of integration and taking matrix elements:

$$\langle f | S | i \rangle = \delta_{p0, p0} \delta_{qf, qf} - \int dx \langle \Phi_{p0}, a_{qf}^{+ \text{out}} i J(x) \Phi_{p0} \rangle f_{qf}(x)$$

We need another reduction formula. We consider:

$$P[\phi(y), J(x)] \equiv \begin{cases} \phi(y) J(x) & \text{for } y_0 > x_0 \\ J(x) \phi(y) & \text{for } x_0 > y_0 \end{cases}$$

We compute:

$$\begin{aligned} & i \int f_{qf}^{+ \text{out}}(y) (\square_y^2 - \mu^2) P[\phi(y), J(x)] dy \\ & \downarrow i f_{qf}^{+ \text{out}}(y) \frac{\partial \phi}{\partial y_0^2} - \phi \frac{\partial^2 f_{qf}^{+ \text{out}}}{\partial y_0^2} \rightarrow \int \frac{\partial}{\partial y_0} i \left(f_{qf}^{+ \text{out}} \frac{\partial \phi}{\partial y_0} - \frac{\partial f_{qf}^{+ \text{out}}}{\partial y_0} \phi \right) = \\ & = \int \frac{\partial}{\partial y_0} P[a_{qf}^{+ \text{out}}(y_0), J(x)] \end{aligned}$$

$$\therefore i \int f_{qf}^{+ \text{out}}(y) (\square_y^2 - \mu^2) P[\phi(y), J(x)] dy = a_{qf}^{+ \text{out}} J(x) - J(x) a_{qf}^{+ \text{in}}$$

Then multiplying this:

$$\begin{aligned} \langle f | S | i \rangle &= \delta_{qf, qf} \delta_{p0, p0} + \int dy f_{qf}^{+ \text{out}}(y) (\square_y^2 - \mu^2) (\Phi_{p0}, P[\phi(y), J(x)] \Phi_{p0}) f_{qf}^{+ \text{out}}(y) dx \\ &\quad - \int dx f_{qf}^{+ \text{out}}(x) (\Phi_{p0}, i J(x) a_{qf}^{+ \text{in}} \Phi_{p0}) \end{aligned}$$

Then finally:

$$\langle p_f q_f | S | p_0 q_0 \rangle = \langle p_f q_f | 1 | p_0 q_0 \rangle + \int f_{qf}^{+ \text{out}}(y) (\square_y^2 - \mu^2) \langle p_f | P[\phi(y), J(x)] | p_0 \rangle f_{qf}^{+ \text{out}}(y) dx$$

Following the conventional field theory, we can now replace the wave packets by plane waves and remove the δ -function singularity by writing:

$$S = \langle p_f q_f | 1 | p_0 q_0 \rangle - (2\pi)^4 i \delta^4(\Delta P_\mu) T$$

$$\begin{aligned} \text{We let: } f_{qf}^{+ \text{out}}(x) &\rightarrow \frac{e^{iq_f x}}{\sqrt{2\omega_0}} \\ f_{qf}^{+ \text{out}}(x) &\rightarrow \frac{e^{-iq_f x}}{\sqrt{2\omega_0}} \end{aligned}$$

Then:

$$-(2\pi)^4 i \delta^4(\Delta P_\mu) T = \frac{1}{\sqrt{4\omega_0 \omega_f}} \int e^{-i q_f y + i q_0 x} dy dx (\square_y^2 - \mu^2) \langle p_f | P[\phi(y), J(x)] | p_0 \rangle$$

We rewrite the exponential as:

$$e^{i(\frac{q_0+q_f}{2})(x-y) + i(q_0-q_f)(\frac{x+y}{2})}$$

To extract the comb of mass dependence of the matrix element we note that

$$\phi(y) = e^{-ip_y y} \phi(0) e^{ip_y y}$$

This is solution of:

$$-\frac{1}{i} \frac{\partial \phi(y)}{\partial y_\mu} = [P_\mu, \phi(y)]$$

(P_μ is the energy-momentum vector). Then:

$$\langle \alpha | \phi(y) | \beta \rangle = \langle \alpha | e^{-i p \cdot y} \phi(0) e^{i p \cdot y} | \beta \rangle = e^{-ip_0 \cdot y + i p_0 \cdot y} \langle \alpha | \phi(0) | \beta \rangle$$

Now consider:

$$\begin{aligned} \langle p_f | \phi(y) J(x) | p_0 \rangle &= \langle p_f | e^{-i p \cdot y} \phi(0) e^{i p \cdot (y-x)} J(0) e^{i p \cdot x} | p_0 \rangle = \\ &= e^{i(p_0 \cdot x - p_f \cdot y)} \langle p_f | \phi(0) e^{i p \cdot (y-x)} J(0) | p_0 \rangle \\ &\downarrow \\ &= e^{i[(p_0 - p_f) \cdot \frac{x+y}{2} + \frac{p_0 + p_f}{2} (x-y)]} \end{aligned}$$

The $\frac{x+y}{2}$ part gives: $e^{i(\frac{x+y}{2}) \cdot (q_0 + p_0 - q_f - p_f)}$

and let: $dx dy = du dw \quad (u = y-x, w = \frac{x+y}{2})$

the integral over dw gives: $(2\pi)^4 \delta^4(q_0 + p_0 - q_f - p_f)$

Then:

$$(2\pi)^4 i T \delta \rightarrow \frac{1}{\int} \int e^{i(\frac{q_0 + q_f}{2}) \cdot (y-x)} F(y-x) d(y-x)$$

We are only motivating what we will do in more detail later.

In the forward direction; the exponential is:

$$e^{-i \vec{q} \cdot \vec{u} + i \omega u_0}$$

and the term $\vec{q} \cdot \vec{u}$ can usually be neglected compared with ωu_0 (e.g. in the atomic case of interactions with light)

Dispersion Relations for Classical light waves.

We consider now $D(t)$ and $E(t)$ (displacement and electric vector) as the fundamental quantities of interest. We have:

$$D(\omega) = E(\omega) E(\omega) \quad \text{for all } \omega.$$

If $E(t) = 0$ for $t < 0$ then $D(t) = 0$ for $t < 0$

Now:

$$E(t) = \int_{-\infty}^{\infty} dw e^{-i \omega t} E(\omega)$$

$$E(\omega) = \frac{1}{2\pi} \int_0^{\infty} e^{i \omega t} dt E(t) \quad (\text{remember } E(t) = 0 \text{ for } t < 0)$$

But now the last expression can be continued into the upper half plane. Let:

$$\omega = \text{Re } \omega + i \text{Im } \omega$$

$$e^{i \omega t} = e^{i \text{Re } \omega t - \text{Im } \omega t}$$

The last exponential makes the integral bounded and the function analytic for $\text{Im } \omega > 0$.

The converse also holds since if $E(\omega)$ is analytic in the upper plane, the integral for $E(t)$ can be closed in the upper plane for $t > 0$ and gives zero.

Now:

$$D(\omega) = \frac{1}{2\pi} \int_0^{\infty} e^{i \omega t} dt D(t)$$

By our causality requirement $D(\omega)$ must be analytic in the upper half plane. But:

$$D(\omega) = \varepsilon(\omega) E(\omega)$$

\downarrow
analytic analytic and arbitrary so we can also choose functions without zeros.

Thus we find:

$$E(\omega) \text{ analytic in } \text{Im } \omega > 0$$

We can now obtain a dispersion relation.



$$\varepsilon(\omega) = \frac{1}{2\pi i} \oint_C \frac{E(\omega')}{\omega - \omega'} d\omega'$$

If $\varepsilon(\omega) \rightarrow 0$ rapidly enough, then we may deform the contour to the real axis. If $\varepsilon(\omega)$ does not approach zero fast enough we consider $\frac{\varepsilon(\omega)}{\omega - \omega_1}$ where $\text{Im } \omega_1 < 0$

Then if we can close contour

$$\varepsilon(\omega + i\epsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega') dw'}{\omega' - \omega - i\epsilon}$$

$$\frac{1}{\omega' - \omega - i\epsilon} = \frac{P}{\omega' - \omega} + i\pi \delta(\omega' - \omega)$$

Then:

$$\varepsilon(\omega) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega') dw'}{\omega' - \omega}$$

Taking the real part:

$$\text{Re } \varepsilon(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im } \varepsilon(\omega') dw'}{\omega' - \omega}$$

For the case when $\varepsilon(\omega) \rightarrow 0$ fast enough, going through the same procedure we obtain: (for $\text{Im } \omega, \omega_1 < 0$)

$$\frac{\varepsilon(\omega)}{\omega - \omega_1} = \frac{P}{\pi i} \int \frac{\varepsilon(\omega') dw'}{(\omega - \omega')(\omega - \omega_1)}$$

We now want to let $\omega_1 \rightarrow$ real axis $\omega_1 = \omega_0 - i\epsilon$

$$\frac{\epsilon(\omega)}{\omega - \omega_0} = \frac{P}{\pi i} \int \frac{\epsilon(\omega') d\omega'}{(\omega' - \omega)} \left[\frac{P}{\omega' - \omega_0} - i\pi \delta(\omega' - \omega_0) \right] =$$

$$= -\frac{\epsilon(\omega_0)}{\omega_0 - \omega} + \frac{P}{\pi i} \int \frac{\epsilon(\omega') d\omega'}{(\omega' - \omega)(\omega' - \omega_0)}$$

$$\therefore \text{Re } \epsilon(\omega) = \text{Re } \epsilon(\omega_0) + \left(\frac{P}{\pi} \int \frac{\text{Im } \epsilon(\omega') d\omega'}{(\omega' - \omega)(\omega' - \omega_0)} \right) (\omega - \omega_0)$$

This result can also be obtained formally by subtracting the less convergent expression for $\text{Re } \epsilon(\omega)$ minus $\text{Re } \epsilon(\omega_0)$, because:

$$\frac{1}{\omega - \omega} - \frac{1}{\omega' - \omega_0} = (\omega - \omega_0) \frac{1}{(\omega - \omega_0)(\omega' - \omega_0)}$$

Dispersion Relations.

Let us return to S matrix.

$$S = 1 + \frac{1}{\sqrt{4w_f w_0}} \int e^{-i\vec{q}_f \cdot \vec{y}} e^{i\vec{q}_0 \cdot \vec{x}} (\Delta^2 - \mu^2) \langle p_f | P[\phi_f(y), J(x)] | p_0 \rangle =$$

$$= 1 + \frac{(2\pi)^4 \delta(p_f + q_f - p_0 - q_0)}{\sqrt{4w_f w_0}} \int e^{-i(\frac{q_f + q_0}{2}) \cdot u} (\Delta^2 - \mu^2) \langle p_f | P[\phi_f(\frac{u}{2}), J_0(-\frac{u}{2})] | p_0 \rangle$$

We would write the integral as: $a(\omega) = \int_0^\infty dt e^{i\omega t} f(t)$

$$-i(\frac{q_f + q_0}{2}) \cdot u = i \left[\left(\frac{w_f + w_0}{2} \right) u_0 - \left(\frac{\vec{q}_f + \vec{q}_0}{2} \right) \cdot \vec{u} \right]$$

at least in

The first part resembles the argument ωt of the above exponential, (the system where the collision is elastic) But the second part is of same order of magnitude than the first and is not negligible.

We need replace the integral for a retarded product. To do this we put the operator $\Delta^2 - \mu^2$ into the bracket. We consider:

$$(\Delta^2 - \mu^2) P[\phi(y), J(x)] \quad \text{and} \quad J(x) = (\Delta^2 - \mu^2) \phi(x)$$

Using the identity: $\epsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$

$$P[\phi(y), J(x)] = \frac{1}{2} \epsilon(y_0 - x_0) [\phi(y), J(x)] + \frac{1}{2} \{ \phi(y), J(x) \}$$

we have:

$$(\Delta^2 - \mu^2) P[\phi(y), J(x)] = P[J(y), J(x)] + \frac{\epsilon}{2} \left[\frac{\partial^2 \phi}{\partial y_0^2}, J \right] - \frac{\partial^2}{\partial y_0^2} \frac{\epsilon}{2} [\phi, J]$$

$$\frac{\partial^2}{\partial y_0^2} \frac{\epsilon}{2} [\phi, J] = \frac{\partial}{\partial y_0} \left\{ \frac{\epsilon}{2} \left[\frac{\partial \phi}{\partial y_0}, J \right] + \frac{1}{2} \frac{\partial \epsilon}{\partial y_0} [\phi, J] \right\} = \frac{\partial}{\partial y_0} \left\{ \frac{\epsilon}{2} \left[\frac{\partial \phi}{\partial y_0}, J \right] \right\} + \delta(x_0 y_0) [\phi, J]$$

In conventional field theory:

$$J(x) = (\Delta^2 - \mu^2) \phi(x) = \sum \bar{p}_p \gamma_p + \lambda \phi^3(x)$$

Then: $[\phi, J] = 0$ In the following we assume valid this result.

In more general theories the constant λ changes very little.

$$\therefore \frac{\partial}{\partial y_0} \left\{ \frac{\epsilon}{2} \left[\frac{\partial \phi}{\partial y_0}, J \right] \right\} = \delta(x_0 y_0) \left[\frac{\partial \phi}{\partial y_0}, J \right] = \frac{1}{x} \delta(x_0 y_0) \times 3 \lambda \phi^2(x) \delta(\vec{x} \cdot \vec{y})$$

$$\therefore (\Delta^2 - \mu^2) \langle p_f | P[\phi(y), J(x)] | p_0 \rangle = -\frac{1}{i} \delta^4(x-y) \langle p_f | 3 \lambda \phi^2(x) | p_0 \rangle + \langle p_f | P[J(y), J(x)] | p_0 \rangle$$

Substituting in the S formula we have:

$$S = 1 + \frac{(2\pi)^4 \delta(\Delta p_k)}{\sqrt{4w_f w_0}} \left[-\frac{1}{i} \langle p_f | 3 \lambda \phi^2(0) | p_0 \rangle + \text{Im } e^{-i(\frac{q_f + q_0}{2}) \cdot u} \langle p_f | P[J(\frac{u}{2}), J(-\frac{u}{2})] | p_0 \rangle \right]$$

The term $\langle p_f | 3 \lambda \phi^2(0) | p_0 \rangle$ is an invariant function $F(p_f, p_0)$. Since both p_f and p_0 have the same magnitude ($p_f^2 = -u^2 = p_0^2$), this function must be depend only on: $\Delta^2 = (p_f - p_0)^2 = -2u^2 - 2p_f \cdot p_0$ (the only invariant)

Then: $F(p_f, p_0) = f(\Delta^2)$ and f is real:

$$f = \langle p_f | 3 \lambda \phi^2(0) | p_0 \rangle = \langle p_0 | 3 \lambda \phi^2(0) | p_f \rangle = f^* \quad (\phi \text{ is selfadjoint})$$

We can now change the over product to a retarded product. Consider the integral:

$$\int e^{-i\vec{q}_f \cdot \vec{y} + i\vec{q}_0 \cdot \vec{x}} \langle p_f | P[J(y), J(x)] | p_0 \rangle$$

We have the identity:

$$P[J(y), J(x)] = \gamma(y_0 - x_0) [J(y), J(x)] + J(x) J(y)$$

$$\text{where } \gamma(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\text{But: } \int e^{-i\vec{q}_f \cdot \vec{y} + i\vec{q}_0 \cdot \vec{x}} \langle p_f | J(x) J(y) | p_0 \rangle = 0$$

since: $\langle p_f | J(x) J(y) | p_0 \rangle = \sum_n \langle p_f | J(x) | n \rangle \langle n | J(y) | p_0 \rangle$ and the integral is:

$$\int e^{i(p_0 - p_n) \cdot y} e^{-i\vec{q}_f \cdot \vec{y}} dy \sim \delta(p_0 - p_n - \vec{q}_f)$$

which is zero because the δ differs of zero only for $p_0 = p_n + \vec{q}_f$. (This implies that we can write (for real physical processes): (nucleon) \rightarrow (meson) + (scattering due). But the

$$S = 1 - \frac{i(2\pi)^4 \delta(\Delta p_k)}{\sqrt{4w_f w_0}} \left| \begin{array}{l} \text{nucleon is stable} \\ \therefore \text{this term must vanish} \end{array} \right.$$

where:

$$T = -f(\Delta^2) + i \int du e^{-i(\frac{q_f + q_0}{2}) \cdot u} \gamma(u_0) \langle p_f | [J_f(\frac{u}{2}), J_0(-\frac{u}{2})] | p_0 \rangle$$

This expression is invariant provided the commutator vanishes for a spacelike separation of the arguments.

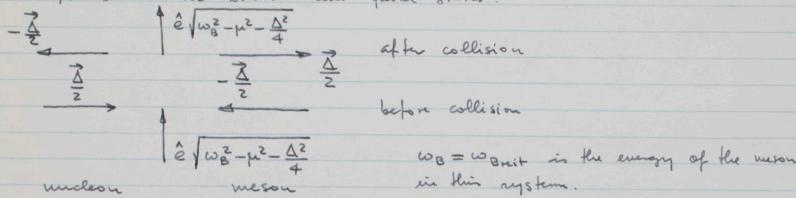
We assume also the validity of the Causality, that is,

$$[J(x), J(y)] = 0 \quad \text{for } (x-y)^2 > 0$$

There is a question whether this is not already assumed when we write down our expression for T and claim that it is invariant.

For these kind of problems, the center of mass system is not a good system.
We choose the Breit coordinate system. (This is the elastic collision system)

It depends on both initial and final states:



We don't worry for isotropic spin and ordinary spin.

The T matrix (with correct sign) is:

$$T = G(\Delta^2) - i \int d\omega y(\omega) e^{-\frac{(q_f+q_0)\cdot u}{2}} \langle p_f | [J(\frac{\omega}{2}), J(-\frac{\omega}{2})] | p_0 \rangle$$

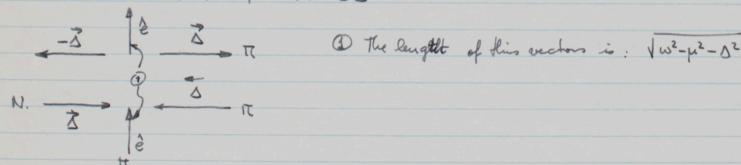
Since $\frac{p_f + p_0}{2} = P$ is a timelike vector, exists a coordinate system in which P has only time part, i.e.: $P = (E_0, 0, 0, 0)$

This is the Breit coordinate system. In this we have:

$$\begin{aligned} p_0 &= \vec{P} \\ p_f &= -\vec{P} \end{aligned}$$

$$E_0 = E_0 = \sqrt{m^2 + \Delta^2} = E_f$$

Note that momentum transfer is $2\vec{\Delta}$



In terms of invariants:

$$v = -\frac{1}{4m} (p_0 + p_f) \cdot (q_0 + q_f) = \frac{E_0 \omega}{m}$$

$$= \frac{E_0}{2m} 2\omega \quad \therefore \omega_B = -\frac{1}{4E_0} (p_0 + p_f) \cdot (q_0 + q_f)$$

Another one invariant is:

$$\Delta^2 = \frac{(q_0 - q_f)^2}{4} = \frac{(p_0 - p_f)^2}{4}$$

The kinematic properties of the system can be easily analyzed in this system. (i.e. parity, time-universal etc.) In B.C.S we have:

$$T = G(\Delta^2) - i \int d\omega \int d\vec{u} e^{i(\omega u - \vec{Q} \cdot \vec{u})} \langle -\vec{\Delta} | [J(\frac{\omega}{2}), J(-\frac{\omega}{2})] | \vec{\Delta} \rangle = T_a + T_b$$

where: $\vec{Q} = \hat{e} \sqrt{\omega^2 - \mu^2 - \Delta^2}$ and $\hat{e} \cdot \vec{u} = 0$ We will consider this without T_a and neglect

Suppose we are dealing with photons (in forward scattering) ($\Delta^2 = 0$)

We call: $w = k$, and then: $\hat{e} \sqrt{\omega^2 - \mu^2 - \Delta^2} = \vec{k}$

The integral is:

$$T_b = -i \int d\omega \int d\vec{u} e^{i(k u - \vec{k} \cdot \vec{u})} \langle \text{rest} | [J(\frac{\omega}{2}), J(-\frac{\omega}{2})] | \text{rest} \rangle$$

We are interested now in the analytic properties of the exponential, for $\Im u > 0$
(It has singularities in the upper half plane). Then for the photons a dispersion relation can be established. (since $\Im u > \Im k$ and $\Im u \omega$ gives damping for $\Im k > 0$)

$$R_T = R_T(k=0) + P \frac{k}{\pi} \int_{-\infty}^{\infty} dk' \frac{\Im T(k')}{k'(k-k')}$$

For vector photons the $R_T(k=0)$ is the Thompson's amplitude $-\frac{e^2}{m}$. For scalar photons this is a constant. (The case μ and $\Delta \neq 0$ gives much more complicated situation). We consider now the quantity:

$$\sqrt{\omega^2 + \alpha^2} = x + iy$$

$$\omega = a + ib$$

$$\omega^2 + \alpha^2 = x^2 - y^2 + 2ixy$$

$$\alpha^2 - b^2 + 2iab + \alpha = x^2 - y^2 + 2ixy$$

$$\therefore ab = xy$$

$$x^2 - y^2 = a^2 - b^2 + \alpha$$

$$\therefore (y^2 - \alpha^2)(1 + \frac{a^2}{y^2}) = -\alpha$$

For $\alpha > 0$ $|y| < |\alpha|$ (the exponential is analytic in the upper half plane)

For $\alpha < 0$ $|y| > |\alpha|$ and is destroyed the analyticity. We also might have difficulty

Note: $T = T(\omega, Q^2, \Delta^2, \vec{Q}, \vec{\Delta})$ (due to the branch point of $\sqrt{\omega^2 - \mu^2 - \Delta^2}$)

This is solved by a trick of Bogoliubov

Related to this: We change $-\mu^2 + \alpha^2 \rightarrow +\alpha$. We need:

$$\lim_{\alpha \rightarrow -\mu^2 + \Delta^2} T(\omega, \Delta^2, \alpha) \quad \text{Then:}$$

$$T_b(\omega, \Delta^2, \alpha) = -i \int d\omega y(\omega) e^{i(\omega u - \vec{Q} \cdot \vec{u})} \langle -\vec{\Delta} | [J(\frac{\omega}{2}), J(-\frac{\omega}{2})] | \vec{\Delta} \rangle$$

$$\vec{Q} = \hat{e} \sqrt{\omega^2 + \alpha^2} \quad (\alpha > 0)$$

But for $\alpha > 0$ this is an analytic function of ω for $\Im \omega$

We can write now a dispersion relation:

$$\text{Re } T_b(\omega, \Delta^2, \alpha) = \frac{P}{\pi} \int_{-\infty}^{\infty} dw' \frac{\text{Im } T_b(w', \Delta^2, \alpha)}{w' - \omega}$$

We want to extend this for $\alpha > \mu^2 - \Delta^2$. We can do this in the real part since only want Re T for $\omega^2 > \mu^2 + \Delta^2$ (physical region). For the values $0 \leq |w| \leq \sqrt{\mu^2 + \Delta^2}$, the exponential has singularities and the formula for T makes no sense. This region is called unphysical region.

We calculate now the imaginary part:

$$2 \text{Im } T_b = - \int_0^\infty du e^{i(u u_0 - Q \cdot \vec{u})} d\vec{u} \langle -\Delta | [J(\frac{u}{2}), J(-\frac{u}{2})] | \Delta \rangle - \\ - i \int_0^\infty du e^{-i(u u_0 - Q \cdot \vec{u})} d\vec{u} \langle \Delta | [J(-\frac{u}{2}), J(\frac{u}{2})] | -\Delta \rangle$$

(because we are dealing with a real neutral field). The matrix element is:

$$\langle \vec{\Delta} | [J(-\frac{u_0}{2}), J(\frac{u_0}{2})] | -\vec{\Delta} \rangle = \langle -\vec{\Delta} | [J(-\frac{u_0}{2}), J(\frac{u_0}{2})] | \vec{\Delta} \rangle$$

using parity conservation. We can also change the sign of Q.

Let $u_0' = -u_0$, the second integral is:

$$-i \int_0^\infty du' e^{i(u u_0' - Q \cdot \vec{u})} \langle -\Delta | [J(\frac{u_0'}{2}), J(-\frac{u_0'}{2})] | \vec{\Delta} \rangle$$

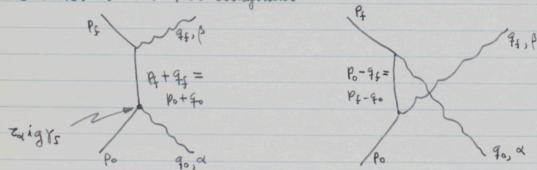
We have then:

$$\text{Im } T_b = -\frac{1}{2} \int_{-\infty}^{\infty} du e^{i(u u_0 - Q \cdot \vec{u})} \langle -\Delta | [J(\frac{u}{2}), J(-\frac{u}{2})] | \Delta \rangle$$

This is the absorptive part (in this sense only real intermediate states can contribute)

Perturbation Theory

We consider the two diagrams:



The second order S matrix is:

$$S = (-i)^2 (2\pi)^4 (2\pi)^4 \delta^4(p_0 + q_0 - p_f - q_f) \cdot \frac{1}{i(2\pi)^4} \frac{1}{i(2\pi)^4} \frac{1}{i(2\pi)^4} \cdot \frac{1}{4\omega_0 \omega_f} \cdot \\ \cdot (\bar{u}_{p_f} [Z_p Z_\alpha] g Y_5 \frac{1}{i\gamma \cdot (p_0 + q_0) + m} ig Y_5 + 2\omega_p ig Y_5 \frac{1}{i\gamma \cdot (p_0 + p_f) + m} ig Y_f) u_{p_0}$$

$$S_F(p) = \frac{1}{(2\pi)^4 i} \frac{1}{i\gamma \cdot p + m} \quad \Delta_F(p) = \frac{1}{(2\pi)^4 i} \frac{1}{p^2 + \mu^2}$$

$$S = -i (2\pi)^4 \frac{\delta(\Delta p_a)}{4\omega_0 \omega_f} T$$

$$T = g^2 (\bar{u}_{p_f} [Z_p Z_\alpha] \frac{Y_5 (i\gamma \cdot (p_0 + q_0) + m) Y_5}{2 p_0 \cdot q_0 - \mu^2} + Z_\alpha Z_p \frac{Y_5 (i\gamma \cdot (p_0 + q_0) + m) Y_5}{-2 p_0 \cdot q_f - \mu^2}) u_{p_0}$$

Our notation is:

$$\{Y_\mu, Y_\nu\} = 2 \delta_{\mu\nu}$$

$$Y_4 = \beta = P_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = -i\vec{\beta}$$

$$\vec{u} = u^+ \beta$$

$$\vec{z} = P_1 \vec{\sigma}$$

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y_5^2 = 1$$

$$Y_5 Y_\mu Y_5 = -Y_\mu$$

$$\text{We can use: } (i\gamma \cdot p_0 + m) u_{p_0} = 0$$

$$\langle i\gamma \cdot (q_0 - q_f) \rangle = \langle i\gamma \cdot (p_0 - p_f) \rangle = 0$$

Thus we have:

$$T = g^2 (\bar{u}_{p_f} [i\gamma \cdot Q] u_{p_0}) \cdot \left[\frac{Z_p Z_\alpha}{2 p_0 \cdot q_0 - \mu^2} - \frac{Z_\alpha Z_p}{-2 p_0 \cdot q_f - \mu^2} \right]$$

$$\text{where: } Q = \frac{q_0 + q_f}{2}$$

We now go to the Breit system to evaluate this:

$$2 p_0 \cdot q_0 = 2 [-E_\Delta \omega - \Delta^2]$$

$$2 p_0 \cdot q_f = 2 [-E_\Delta \omega + \Delta^2]$$

Then:

$$T = g^2 (\bar{u}_{p_f} i\gamma \cdot Q u_{p_0}) \left[\frac{Z_p Z_\alpha}{-2 E_\Delta \omega - 2 \Delta^2 - \mu^2} - \frac{Z_\alpha Z_p}{2 E_\Delta \omega - 2 \Delta^2 - \mu^2} \right]$$

Now we can write the wave function:

$$u(p) = (1 + \frac{\vec{z} \cdot \vec{p}}{m + \epsilon}) \begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} \equiv 1 + \frac{\vec{z} \cdot \vec{p}}{m + \epsilon}$$

We must normalize this:

$$(u(p), u(p)) = (1 + \frac{\vec{z} \cdot \vec{p}}{m + \epsilon})^2 = 1 + \frac{p^2}{(m + \epsilon)^2} + \frac{2 \vec{z} \cdot \vec{p}}{m + \epsilon} \\ = \frac{m^2 + 2m\epsilon + \epsilon^2 + p^2}{(m + \epsilon)^2} = \frac{2\epsilon}{m + \epsilon}$$

Then we can calculate:

$$(\bar{u}_{p_f} i\gamma \cdot Q u_{p_0}) = (u_{-\Delta}, \beta i\gamma \cdot Q u_\Delta) = \begin{pmatrix} \frac{Q_0 = \omega}{Q = \sqrt{\omega^2 - \mu^2 - \Delta^2}} \\ \frac{Q_0 = \omega}{Q = \sqrt{\omega^2 - \mu^2 - \Delta^2}} \end{pmatrix} \\ = (1 + \frac{\vec{z} \cdot \vec{Q}}{m + \epsilon}) (\underbrace{p_f p_i \omega + \beta \cdot (-i\vec{p} \cdot \vec{z}) \cdot \vec{Q}}_{-w + \vec{z} \cdot \vec{Q}}) (1 + \frac{\vec{z} \cdot \vec{Q}}{m + \epsilon}) \frac{m + \epsilon}{2 E_\Delta} =$$

$$= -\omega \left(1 - \frac{\Delta^2}{(m+E_\Delta)^2} \right) \frac{m+E_\Delta}{2E_\Delta} + \frac{m+E_\Delta}{2E_\Delta} \frac{1}{(m+E_\Delta)} (-2i\vec{Q} \cdot \vec{\Delta} \times \vec{Q}) =$$

$$= -\frac{\omega m}{E_\Delta} - i \frac{\vec{Q} \cdot \vec{\Delta} \times \vec{Q}}{E_\Delta}$$

since: $\vec{Q} \cdot \vec{\Delta} \vec{Q} = \vec{p}_f \cdot \vec{\Delta} \vec{p}_i \vec{p}_i \cdot \vec{Q} = \vec{p}_i \cdot \vec{\Delta} \vec{p}_i \vec{Q} = \vec{\Delta} \vec{Q} + i \vec{p}_i \cdot \vec{\Delta} \times \vec{Q}$

Thus we have:

$$T = \left(\frac{\omega m}{E_\Delta} + i \frac{\vec{Q} \cdot \vec{\Delta} \times \vec{Q}}{E_\Delta} \right) \left(\frac{Z_\Delta Z_\alpha}{2\Delta^2 + \mu^2 + 2E_\Delta \omega} + \frac{Z_\alpha Z_\Delta}{-(2\Delta^2 + \mu^2) + 2E_\Delta \omega} \right) g^2$$

Now:

$$1) \quad \vec{\Delta} \times \vec{Q} = \frac{\vec{q}_f - \vec{q}_i}{2} \times \frac{\vec{q}_f + \vec{q}_i}{2} = \frac{\vec{q}_f \times \vec{q}_i}{2}$$

2) real

$$3) \text{ poles at: } a) \frac{1}{2E_\Delta(\omega - \omega_B)} \quad \omega_B = -\frac{\Delta^2 + \mu^2}{2E_\Delta} \approx \mu \left(\frac{m}{\mu} \right)$$

$$b) \frac{1}{2E_\Delta(\omega + \omega_B)}$$

4) Crossing symmetry:

$$\omega \rightarrow -\omega \quad \vec{Q} \rightarrow -\vec{Q} \quad \alpha \leftrightarrow \beta$$

T is invariant under this transformation.

We can formulate the dispersion relation: ($\alpha > 0$)

$$\text{Re } T(\omega, \Delta^2, \alpha) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{dw'}{\omega' - \omega} \text{Im } T(w')$$

where:

$$T = -i \int e^{i[\omega u_0 - \hat{e}\sqrt{\omega^2 + \alpha} \cdot \vec{u}]} du_0 d\vec{u} \delta(u_0) \langle -\vec{\Delta} | [J(\frac{u}{2}), J(-\frac{u}{2})] | \vec{\Delta} \rangle$$

and $\hat{e} \cdot \vec{\Delta} = 0$, $\vec{Q} = \hat{e} \sqrt{\omega^2 + \alpha}$. We want consider and for the actual problem $\omega \rightarrow -\mu^2 - \Delta^2$

Consider now the absorptive part: (remember that we have neglected charge, spin, etc)

$$\text{Im } T = -\frac{1}{2} \int_{-\infty}^{\infty} e^{i[\omega u_0 - \vec{Q} \cdot \vec{u}]} du_0 d\vec{u} \langle -\vec{\Delta} | [J(\frac{u}{2}), J(-\frac{u}{2})] | \vec{\Delta} \rangle$$

one-particle P_f P_i

Using a complete set of intermediate states, the bracket is:

$$\sum_n \langle -\Delta | J | n \rangle e^{i(P_f - P_i) \frac{u}{2}} \langle n | J | \Delta \rangle e^{-i(P_f - P_i) \frac{u}{2}} - \text{the same expression with } u \rightarrow -u$$

$$= \sum_n \langle -\Delta | J | n \rangle \langle n | J | \Delta \rangle [e^{i(P_f - \frac{P_0 + P_f}{2}) \cdot u} - e^{-i(P_f - \frac{P_0 + P_f}{2}) \cdot u}]$$

The sum is really an integral because the one-particle intermediate states ~~must~~ belong to continuum.

Since the first term of the square bracket can be written as:
 $e^{-i(E_n - E_\Delta) u_0 + i \vec{p}_n \cdot \vec{u}}$

$$\text{we have: } \int e^{i\omega u_0 - i \vec{Q} \cdot \vec{u} - i(E_n - E_\Delta) u_0 + i \vec{p}_n \cdot \vec{u}} du_0 d\vec{u} = (2\pi)^3 \delta(\vec{p}_n - \vec{Q})(2\pi) \delta[\omega - (E_n - E_\Delta)]$$

Similarly for the second term we have:

$$-(2\pi)^3 \delta(\vec{p}_n + \vec{Q})(2\pi) \delta[\omega + (E_n - E_\Delta)]$$

We can change $\delta(\vec{p}_n + \vec{Q}) \rightarrow \delta(\vec{p}_n - \vec{Q})$ because the sign of \vec{Q} is unimportant for the intermediate states. Thus we have:

$$\text{Im } T = -\frac{1}{2} (2\pi)^4 \sum_n \langle -\Delta | J | n \rangle \langle n | J | \Delta \rangle \delta(\vec{p}_n - \vec{Q}) [\delta(\omega - (E_n - E_\Delta)) - \delta(\omega + (E_n - E_\Delta))]$$

Since this is an odd function of ω , we have:

$$\text{Im } T(\omega) = -\text{Im } T(-\omega)$$

and then $\text{Re } T$ is even.

What are the intermediate states?

Obviously they depend on the process considered. We consider only nucleon-meson scattering. We choose these states as:

- | | |
|-------------------------|----------|
| 1) 1 nucleon | mass |
| 2) 1 nucleon + 1 meson | m |
| 3) 1 nucleon + 2 mesons | $>m+\mu$ |
| ----- | ----- |

We define the mass of the intermediate state as:

$$E_n^2 - P_n^2 \equiv M_n^2$$

We'll consider only the 1) in detail. Note that this gives to a no observable cross-section. For these states (remember we neglect spin and charge for the moment) the completeness relation gives:

$$\sum_n \rightarrow \int \frac{d\vec{p}}{(2\pi)^3} \Phi_p \rangle \langle \Phi_p \quad (\vec{p} \text{ is the nucleon moment})$$

Then:

$$\text{Im } T_{\text{nucle.}} = -\frac{1}{2} (2\pi) \langle -\vec{\Delta} | J | \vec{Q} \rangle \langle \vec{Q} | J | \vec{\Delta} \rangle \delta[\omega - (E_n - E_\Delta)] - \text{crossed term}$$

$$\text{But } \langle p_f | J | p_i \rangle = \frac{1}{\sqrt{2E_\Delta}} \frac{1}{\sqrt{2E_\Delta}} f[(p_f - p_i)^2]$$

$$\text{and: } (p_f - p_i)^2 = (-\vec{Q} \cdot \vec{Q})^2 - (E_\Delta - E_\Delta)^2 = \Delta^2 + Q^2 - \omega^2$$

$$\therefore (\Delta p)^2 = (\Delta^2 + \alpha)$$

$$\text{and } \langle p_f | J | p_i \rangle = \frac{1}{\sqrt{2E_a}} \frac{1}{\sqrt{2E_b}} f(\Delta^2 + \alpha)$$

For $\alpha = -\mu^2 - \Delta^2$, $f \rightarrow f(-\mu^2)$ which correspond to no physical process.
We have then:

$$\text{Im } T_{\text{inel}} = -\pi \frac{1}{2E_a} \frac{1}{2E_b} [f(\Delta^2 + \alpha)]^2 \delta(\omega - (E_b - E_a))$$

The argument of δ is:

$$\begin{aligned} \omega - (\sqrt{\omega^2 + Q^2} - \sqrt{\mu^2 + \Delta^2}) &= \frac{(\omega + E_a - E_b)(\omega + E_a + E_b)}{\omega + E_a + E_b} = \frac{(\omega + E_a)^2 - (\omega^2 + \Delta^2) - \mu^2}{2E_a} \\ &= \frac{2\omega E_a + \Delta^2 - \alpha}{2E_a} \end{aligned}$$

$$\begin{aligned} \therefore \text{Im } T_{\text{inel}} &= -\pi \frac{1}{2E_a 2E_b} f^2 2E_b \delta(2\omega E_a + \Delta^2 - \alpha) = \\ &= -\frac{\pi}{(2E_a)^2} [f(\Delta^2 + \alpha)]^2 \delta(\omega - \omega_B) \end{aligned}$$

$$\text{where } \omega_B \equiv \frac{\alpha - \Delta^2}{2E_a} \quad B = \text{bound.}$$

Then: $\text{Re } T = \frac{P}{\pi} \int \frac{d\omega' d\omega}{\omega' - \omega} \text{Im } T$ contains contributions only for $\omega = \omega_B$

For the case: $\alpha \rightarrow -\mu^2 - \Delta^2$ (We will show later that we can take this limit)

$$\text{Im } T_{\text{inel}} \rightarrow -\frac{\pi}{(2E_a)^2} f^2 (-\mu^2) \delta(\omega - \omega_B)$$

$$\text{and: } \omega_B = -\frac{\mu^2 + 2\Delta^2}{2E_a}$$

We found:

$$\text{Im } T_{\text{bound state}} = -\frac{\pi}{(2E_a)^2} [f(\Delta^2 + \alpha)]^2 [\delta(\omega - \omega_B) - \delta(\omega + \omega_B)]$$

This is for the cross term.

Thus, the contribution of the bound state is:

$$\begin{aligned} \text{Re } T_{\text{bound state}} &= \frac{P}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\text{Im } T(\omega)}{\omega' - \omega} = -\frac{[f(\Delta^2 + \alpha)]^2}{(2E_a)^2} \left[\frac{1}{\omega_B - \omega} + \frac{1}{\omega_B + \omega} \right] = \\ &= -\frac{2\omega_B}{(2E_a)^2} \frac{[f(\Delta^2 + \alpha)]^2}{\omega_B^2 - \omega^2} \end{aligned}$$

Comparing with perturbative theory we recognize:

$$\begin{aligned} 1) \quad \langle \text{inel} | J(x) | \text{inel} + n \text{pions} \rangle &\sim g^2 \text{ or greater} \\ \therefore \text{Im } T &\sim g^4 \end{aligned}$$

2) To lowest order $f(y) = g$ independent of y .

Then:

$$\text{Re } T_{\text{bound state}} (P.T.) = -\frac{2\omega_B g^2}{(2E_a)^2 (\omega_B^2 - \omega^2)}$$

(Note that it has the same structure as our direct perturbation theory calculation) and we have for all cases:

$$\frac{\omega - \omega_B^2 + i\omega_B^2}{\omega^2 - \omega_B^2} = 1 + \frac{1}{\omega^2 - \omega_B^2}$$

Now we consider the rest of the imaginary part.

$$\text{Im } T = -\frac{(2\pi)^4}{2} \sum_n \langle -\Delta | J | Q_n \rangle \langle Q_n | J | \Delta \rangle [\delta(\sqrt{M_n^2 + Q_n^2} - E_a - \omega) - \delta(\sqrt{M_n^2 + Q_n^2} - E_a + \omega)]$$

Note that $M_n^2 \geq (m + \mu)^2$ (There are no bound states and we have eliminated the bare term)

From the argument of the 1st δ we have as permitted values of ω :

$$\sqrt{M_n^2 + Q_n^2} = E_a + \omega$$

$$\therefore M_n^2 + \omega^2 + \alpha = \omega^2 + \Delta^2 + \omega^2 + 2E_a \omega$$

$$2E_a \omega = M_n^2 + \alpha - \omega^2 - \Delta^2$$

The minimum value of ω that permits the 1st δ is:

$$\omega_{\min} = \frac{(m + \mu)^2 + \alpha - \omega^2 - \Delta^2}{2E_a}$$

Then we have the dispersion relation: $\text{Re } T = \frac{A}{\omega - \omega_{\min}} + \frac{B}{\omega + \omega_{\min}}$

For the finite momentum transfer (and $\alpha \rightarrow -\mu^2 - \Delta^2$) see: Lehman N.C. (last number)
Dyson P.R. (spring 1958), Jost and Lehman N.C. (2 years ago). All these papers are very complicated mathematically. In our case:

$$\omega_{\min} = \frac{\omega^2 + 2\omega_E \alpha + \mu^2 - \mu^2 - \Delta^2 - \omega^2 - \Delta^2}{2E_a} = \frac{m\mu - \Delta^2}{2E_a}$$

$$\text{and: } Q_{\min}^2 = \omega_{\min}^2 + \alpha$$

$$\text{For: } \alpha \rightarrow -\mu^2 - \Delta^2 \quad \omega \rightarrow \frac{m\mu - \Delta^2}{E_a}$$

$$Q_{\min}^2 \rightarrow -\frac{\Delta^2(m + \mu)^2}{E_a^2}$$

Then for $\Delta^2 = 0$ $\omega \rightarrow \mu$ $Q_{\min}^2 \rightarrow 0$ and the relations are proved in this case. The proof fails for $\Delta^2 \neq 0$. One can also consider derivatives of T with respect to Δ^2 at $\Delta^2 = 0$. Then we have:

$$\text{Re } T(\omega, \alpha, \Delta^2) = \frac{[f(\alpha + \Delta^2)]^2}{(2E_a)^2} \left[\frac{1}{\omega_B - \omega} + \frac{1}{\omega_B + \omega} \right] +$$

$$+ \frac{P}{\pi} \int_{-\infty}^{\infty} d\omega' \text{Im } T(\omega, \Delta^2, \alpha) \left[\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right]$$

$$\text{with: } 2E_a \omega' = M^2 + \alpha - \omega^2 - \Delta^2$$

We take $\alpha = -(\mu^2 + \Delta^2)\beta$ and then limit $\beta \rightarrow 1$
(as redefining by one of the total C.F. Mass energy M)

Substituting $\omega' \rightarrow M'^2$ in the second integral we have:

$$\int_{M'^2}^{\infty} dM'^2 \operatorname{Im} T(M'^2, \Delta^2, \alpha) \left[\frac{1}{M'^2 - M^2} + \frac{1}{M'^2 + M^2 + 2\mu^2 - 2\Delta^2 - 2\mu\epsilon} \right]$$

There is now no dependence on lower limit or on the principal value integral.

We can also differentiate $\operatorname{Im} T$ as often as we like since all threshold singularities occur in the M'^2 variable.

Born term is also OK since:

$$f^2(\Delta^2) = f^2(\Delta^2(1-\beta) - \mu^2\beta)$$

evaluate at $\beta=1$ all derivatives OK. Also we can take limit $\Delta \rightarrow -\mu^2 - \Delta^2$ in f^2 and is real since everything else is.

$f^2(-\mu^2) = g^2$ is unchanged at large distances

We have derived:

$$\operatorname{Re} T(\omega, \Delta^2) = \frac{A}{\omega_B^2 - \omega^2} + \frac{P}{\pi} \int_{\omega_{\min}}^{\infty} \frac{2\omega' d\omega'}{\omega'^2 - \omega^2} \operatorname{Im} T(\omega', \Delta^2)$$

$$\omega_{\min} = \frac{m\mu - \Delta^2}{E_\Delta}$$

for an infinitesimal region about $\Delta = 0$

For finite Δ^2 there is an unphysical region

$$\omega_{\min} \leq \omega \leq \sqrt{\mu^2 + \Delta^2} \quad \text{or} \quad \cos\theta' < -1$$

In this region we must also learn how evaluate $\operatorname{Im} T$ in the unphysical region Lehmann has shown that Legendre expansion is valid and a Disp. rel. can be proven for:

$$\Delta^2 < \frac{B}{3} \frac{2\mu + \mu}{2\mu - \mu}$$

We can change the above Disp. Rel. to:

$$T(\omega, \Delta^2) = \frac{A}{\omega_B^2 - \omega^2} + \frac{1}{\pi} \int_{\omega'^2 - \omega^2 - i\epsilon}^{\infty} \operatorname{Im} T(\omega', \Delta^2)$$

The imaginary part of this equation is an identity.

We must remember that we might have to make a subtraction from this equation if the integral does not converge.

In the forward direction:

$\Delta^2 = 0$ Breit system \rightarrow Lab. syst.

$$T = \frac{A}{\omega_B^2 - \omega^2} + \frac{1}{\pi} \int \frac{d\omega'^2}{\omega'^2 - \omega^2} \operatorname{Im} T(\omega')$$

But

$$\operatorname{Im} T(\omega') = q^1 \frac{\operatorname{Tr}(T(\omega'))}{4\pi} \quad \text{where } q^1 \text{ is the momentum in the lab. system}$$

Then our integral goes to:

$$\int \frac{d\omega'^2}{\omega'^2 - \omega^2} q^1 \operatorname{Tr}(T(\omega'))$$

If we assume $\operatorname{Tr}(T(\omega')) \rightarrow \text{const.}$ then the integral diverges and we must make a subtraction. This is not too bad since we already have a unknown $T - T'$ constant to determine.

We now go back and start over including charge and spin.
Now we had:

$$T_{\beta\beta; \mu\mu} = -i \int e^{-i\vec{q}_f \cdot \vec{x}} e^{i\vec{q}_f \cdot \vec{y}} dx dy (D_x^2 - \mu^2) \langle p_f | \gamma^\mu | \gamma(x), J_\mu(y) \rangle | p_i, \nu \rangle$$

Now we have seen that we can write:

$$T_{\mu\mu} = \langle \infty | \{ \exp [i t_1 + i \sigma \cdot \vec{q}_f \times \vec{q}_0 t_2] + [z_\beta, z_\alpha] [t_3 + i \sigma \cdot \vec{q}_f \times \vec{q}_0 t_4] \} | \infty \rangle$$

where: $t_i = t_i(\theta, \omega)$

We now want to write down Disp. rel. for t_1, t_2, t_3, t_4 .

We must separate each t_i into dispersion and absorption part. We claim that this is done and that by our choice:

$\operatorname{Im} T_i = \text{Absorptive part of } t_i$

$\operatorname{Re} T_i = \text{Dispersion part of } t_i$

Also these must have the subtractive terms real.

We should not be surprised since we note that these amplitudes are in terms of:

$$\frac{e^{i\delta} \sin \delta}{q} \rightarrow \Gamma$$

We will work out the Disp. Rel. for t_1 and
Problem 4. Work out the Disp. Rel. for t_2 .

To project out t_1 we must compute:

$$t_1 = \frac{1}{2} \frac{1}{3} \sum_{\mu\mu} T_{\mu\mu; \mu\mu} = \text{Ave. } T$$

To project out t_3 we compute:

$$t_3 = \frac{1}{N} \frac{1}{2} \sum_{\beta\beta} \sum_{\mu\mu} T_{\beta\beta; \mu\mu; \beta\beta} [z_\beta, z_\alpha]_{\beta\beta}$$

under charge.

$$N = \sum_{\beta\beta} [z_\beta, z_\alpha]_{\beta\beta}, [z_\beta, z_\alpha]_{\beta\beta} = \operatorname{Tr} [z_\beta, z_\alpha]^2$$

To project out t_2

$$\begin{aligned} (\vec{q}_f \times \vec{q}_0)^2 t_2 &= \operatorname{Tr} [i \sigma \cdot \vec{q}_f \times \vec{q}_0]_{\mu\mu} T_{\mu\mu} = \\ &= \vec{q}_f \cdot \vec{q}_0 \times (\vec{q}_f \times \vec{q}_0) t_2 = \vec{q}_f \cdot [\vec{q}_0^2 \vec{q}_f - (\vec{q}_0 \cdot \vec{q}_f) \vec{q}_0] t_2 \\ &= [\vec{q}_0^2 \vec{q}_f^2 - (\vec{q}_0 \cdot \vec{q}_f)^2] t_2 \end{aligned}$$

We want to project out T_2 .

$$\begin{aligned} T_{1m,2p,q'q} &= \delta_{qp} [T_1 + i(\vec{q}_f \times \vec{q}_o) \cdot \vec{T}_2]_{m'm, q'q} \\ &\downarrow \\ -2i\vec{\sigma} \cdot \vec{Q} \times \vec{\Delta} \\ \text{Tr}(\vec{\sigma} \cdot \vec{Q} \times \vec{\Delta})(\vec{\sigma} \cdot \vec{Q} \times \vec{\Delta}) &= \vec{Q} \times \vec{\Delta} \cdot \vec{Q} \times \vec{\Delta} = \vec{Q} \cdot [\vec{\Delta} \times (\vec{Q} \times \vec{\Delta})] = \\ &= \vec{Q} \cdot \vec{\Delta}^2 \vec{\Delta} = Q^2 \Delta^2 \quad \vec{Q} = \frac{\vec{q}_f + \vec{q}_o}{2} \\ \text{Thus:} \quad Q^2 \Delta^2 T_2 &= \text{Tr } \vec{\sigma} \cdot \vec{Q} \times \vec{\Delta} T \quad \vec{\Delta} = \frac{\vec{q}_f - \vec{q}_o}{2} \end{aligned}$$

To use this we use the representation

$$T = -i \int e^{i(w_0 - \vec{Q} \cdot \vec{u})} du d\vec{u} \langle u | \dots | u \rangle$$

Now in this relation:

$$\vec{\sigma} \times \vec{\Delta} \cdot \vec{Q} = \vec{\sigma} \times \vec{\Delta} \cdot \frac{\vec{\nabla} u}{i}$$

We now look at the additive constant.

$$\text{Now: } T_1 = A u T$$

$$T_1 = A u (-i) \int e^{-i(\vec{q}_f \cdot \vec{y} + \vec{q}_o \cdot \vec{x})} d^4 y d^4 x (\Box_y^2 - \mu^2) y(y_0 - x_0) \langle p_{f,h}^m | [\phi_h(y), J_\alpha(x)] | p_{o,h}^m \rangle$$

We want:

$$\begin{aligned} &- \frac{\partial^2}{\partial y_0^2} \vec{q}_f + \gamma \frac{\partial^2}{\partial y_0^2} \phi \\ &- \frac{\partial}{\partial y_0} \left(\frac{\partial y}{\partial y_0} \phi + \gamma \frac{\partial \phi}{\partial y_0} \right) + \gamma \frac{\partial^2}{\partial y_0^2} \phi \\ &- \frac{\partial^2 \eta}{\partial y_0^2} \phi - 2 \frac{\partial y}{\partial y_0} \frac{\partial \phi}{\partial y_0} \end{aligned}$$

Integrate this by parts since δ -function makes everything vanish

$$+ \frac{\partial y}{\partial y_0} \frac{\partial \phi}{\partial y_0} - 2 \frac{\partial y}{\partial y_0} \frac{\partial \phi}{\partial y_0} + \frac{\partial y}{\partial y_0} \phi \cdot i w$$

Then:

$$\begin{aligned} T_1 &= A u (-i) \int e^{-i(\vec{Q} \cdot \vec{u})} \{ \langle u, p_f^m | -[\pi_\alpha(\frac{\vec{u}}{2}), J_\alpha(-\frac{\vec{u}}{2})] | u, \vec{\Delta} \rangle + \\ &+ i w \langle u, -\vec{\Delta} | [\phi(\frac{\vec{u}}{2}), J_\alpha(-\frac{\vec{u}}{2})] | u, \vec{\Delta} \rangle \} \end{aligned}$$

$\hookrightarrow 0$ by time reversal invariance.

Under time reversal:

$$\begin{aligned} \phi &\rightarrow -\phi \\ \vec{J} &\rightarrow -\vec{J} \quad [\dots] \rightarrow +[\dots] \end{aligned}$$

$\vec{\Delta} \leftrightarrow -\vec{\Delta}$ m-states mixed but average is not changed

$$\sum_m \rightarrow \sum_m$$

$$\begin{aligned} < \dots > \rightarrow < \dots >^* \\ < \dots > \rightarrow \langle u, \vec{\Delta} | [\phi, J] | u, -\vec{\Delta} \rangle^* &= \langle u, -\vec{\Delta} | [\phi, J]^+ | u, \vec{\Delta} \rangle = \\ &= -\langle u, -\vec{\Delta} | [\phi, J] | u, \vec{\Delta} \rangle \end{aligned}$$

This does not work for first term since π_α has opposite properties under time reversal as ϕ .

We will now show that the first term (which we call $A(\Delta^2, Q^2)$) is real. We compute:

$$A^*(\Delta^2, Q^2) = i \text{Av} \left[e^{i(\vec{Q} \cdot \vec{u})} \langle u, \vec{\Delta} | [\pi_\alpha(\frac{\vec{u}}{2}), J_\alpha(-\frac{\vec{u}}{2})] | u, -\vec{\Delta} \rangle \right]$$

But we can change sign of \vec{Q} and $\vec{\Delta}$ since it is a function only of Q^2 and Δ^2 . Thus:

$$A^*(\Delta^2, Q^2) = A(\Delta^2, Q^2)$$

By causality $[\dots]$ must vanish for $\vec{u} > 0$. Thus it must contain only a finite number of derivatives of $\delta(\vec{u})$

$$\delta(\vec{u}) \quad \nabla^2 \delta(\vec{u}) \quad \dots$$

$\downarrow \quad \downarrow \quad \downarrow \quad \dots$ with coefficients that are arbitrary functions of Δ^2

If had infinite number there it would be non-local which is impossible. Only powers of Q^2 appear.

We now calculate the bound state term.

$$\begin{aligned} \text{Im } T &= -\frac{\text{Av}}{2} \int d^4 u e^{i(w_0 - \vec{Q} \cdot \vec{u})} \langle u, -\vec{\Delta} | [J_\alpha(\frac{\vec{u}}{2}), J_\alpha(-\frac{\vec{u}}{2})] | u, \vec{\Delta} \rangle = \\ &= -\frac{\text{Av}}{2} \cdot \frac{(2\pi)^4}{(2\pi)^4} S(w + E_\Delta - E_Q) \langle u, -\vec{\Delta} | J_\alpha(0) | \vec{Q} \rangle \langle \vec{Q} | J_\alpha(0) | u, \vec{\Delta} \rangle = \\ &= -\pi S(w - w_B) \cdot \frac{2E_Q}{2E_\Delta} g^2 \text{Av } M \end{aligned}$$

where $M = (\bar{u}(-\Delta), i\gamma_5 u(\vec{Q})) (\bar{u}(\vec{Q}), i\gamma_5 u(\vec{\Delta}))$

Since by invariance arguments

$$\langle p_2 | J_\alpha | p_1 \rangle = \bar{u}(p_2) i\gamma_5 \gamma_\alpha u(p_1) F((4p)^2) \xrightarrow{-\mu^2}$$

and we define $g = F((4p)^2 - \mu^2)$

$$\bar{u} = (u, \beta) \quad \beta = \beta \cdot \gamma_5 = \gamma_1 \quad \vec{\Delta} = p_1 \vec{P}$$

$$M \rightarrow (u(-\vec{\Delta}), p_1 \Lambda^+(\vec{Q}) p_1 u(\vec{\Delta}))$$

$$\Lambda^+(\vec{Q}) = \frac{H(\vec{Q}) + E(\vec{Q})}{2E(\vec{Q})}$$

$$\begin{aligned}
M &\rightarrow \left(1 - \frac{\vec{z} \cdot \vec{\Delta}}{m + E_\Delta}\right) p_1 \left(\frac{\vec{z} \cdot \vec{Q} - \beta m + \bar{E}_Q}{2\bar{E}_Q}\right) p_1 \left(1 + \frac{\vec{z} \cdot \vec{\Delta}}{m + E_\Delta}\right) \frac{1}{1 + \frac{\Delta^2}{(m + E_\Delta)^2}} = \\
&= \left(\frac{E_Q - M}{2\bar{E}_Q} - \frac{E_Q + M}{2\bar{E}_Q} \frac{\Delta^2}{(m + E_\Delta)^2}\right) \frac{1}{1 + \frac{\Delta^2}{(m + E_\Delta)^2}} = \\
&= \frac{1}{2\bar{E}_Q \left(1 + \frac{\Delta^2}{(m + E_\Delta)^2}\right)} \left\{ E_Q \left[1 - \frac{\Delta^2}{(m + E_\Delta)^2}\right] - M \left[1 + \frac{\Delta^2}{(m + E_\Delta)^2}\right]\right\} = \\
&= \frac{2M(m + E_\Delta)}{2\bar{E}_Q \frac{2(M + E_\Delta)E_\Delta}{(m + E_\Delta)^2}} [E_Q - E_\Delta] = \frac{1}{2} \frac{m}{E_\Delta \bar{E}_Q} \omega_B
\end{aligned}$$

Thus:

$$\text{Im } T_1 = -\pi \delta(\omega - \omega_B) \frac{g^2 m \omega_B}{2 E_\Delta} + \text{cross term.}$$

We have found:

$$T = S_{\vec{q}, \vec{p}} [T_1 + i \vec{P} \cdot \vec{\Delta} \times \vec{Z} T_2] + \frac{[Z_\beta, Z_\alpha]}{2} [T_3 + i \vec{P} \cdot \vec{\Delta} \times \vec{Z} T_4]$$

With the dispersion relation for T_1 : $\int_{\omega_{\min}}^{\infty} \frac{2\omega' d\omega'}{\omega'^2 - \omega^2} \text{Im } T_1(\omega')$ (in Breit coordinate system)

$$\text{Re } T_1 = \frac{g^2 m}{E_\Delta^2} \frac{\omega^2}{\omega - \omega_B^2} + \frac{P}{\pi} \int_{\omega_{\min}}^{\infty} \frac{2\omega' d\omega'}{\omega'^2 - \omega^2} \text{Im } T_1(\omega')$$

$$\text{where } \omega_B = -\frac{(m + 2\Delta)}{E_\Delta} \quad \omega_{\min} = \frac{m}{E_\Delta} \left(m - \frac{\Delta^2}{m}\right)$$

and we have similar expressions for T_2 and T_3 and T_4 . Note that T_1 and T_4 are even. The equivalent result in perturbation theory is: (and T_2 and T_3 odd functions).

$$T = \frac{g^2 m}{2 E_\Delta^2} \left\{ \left[\omega + i \vec{P} \cdot \frac{\vec{\Delta} \times \vec{\alpha}}{m}\right] \frac{Z_\beta Z_\alpha}{\omega - \omega_B} + \left[\omega + i \vec{P} \cdot \frac{\vec{\Delta} \times \vec{\alpha}}{m}\right] \frac{Z_\alpha Z_\beta}{\omega + \omega_B} \right\}$$

Note that:

$$\text{even } \frac{1}{\omega' - \omega} + \frac{1}{\omega + \omega} = \frac{2\omega'}{\omega'^2 - \omega^2}$$

$$\text{odd } \frac{1}{\omega' - \omega} - \frac{1}{\omega + \omega} = \frac{2\omega}{\omega'^2 - \omega^2}$$

$$\text{Then: } \text{Re } T_1 = \frac{g^2 m \omega^2}{E_\Delta^2 (\omega^2 - \omega_B^2)} + \int 2\omega'$$

$$\text{Re } T_2 = \frac{g^2}{E_\Delta^2 (\omega^2 - \omega_B^2)} \omega + \int 2\omega$$

$$\text{Re } T_3 = \frac{g^2}{E_\Delta^2} \frac{m \omega \omega_B}{\omega^2 - \omega_B^2} + \int 2\omega$$

$$\text{Re } T_4 = \frac{g^2 \omega_B}{E_\Delta^2 (\omega^2 - \omega_B^2)} + \int 2\omega'$$

We consider, in the next, the following points:

1. The high energy dependence of T and the number of subtractions that are necessary. In particular we will see the convenient arguments of Pomeranchuk to show that T_3 needs no subtractions.
2. The experimental verification of the forward direction of T_1 and T_3 .
3. The prediction on P -wave amplitudes.

At high energies:

$$\omega_B = -\frac{\mu^2}{2m} \quad \omega_{\min} = M$$

$$\therefore \frac{\omega_B^2}{\mu^2} \sim \left(\frac{M}{m}\right)^2 \lesssim \frac{1}{150}$$

Then the ω_B can be neglected. Then:

$$\text{Re } T_1 = \frac{g^2}{m} + \frac{P}{\pi} \int \frac{d\omega' 2\omega'}{\omega'^2 - \omega^2} \text{Im } T_1$$

$$\text{Re } T_3 = -\frac{g^2}{2m^2} \frac{1}{\omega} + \frac{P}{\pi} \int \frac{d\omega' 2\omega}{\omega'^2 - \omega^2} \text{Im } T_3$$

$$\text{and: } T = S_{\vec{q}, \vec{p}} T_1 + \frac{[Z_\beta, Z_\alpha]}{2} T_3$$

$$\beta = \frac{1-i\varepsilon}{\sqrt{2}} \quad \alpha = \frac{1+i\varepsilon}{\sqrt{2}}$$

$$Z_X \rightarrow \frac{Z_1 + i Z_2}{\sqrt{2}} = \sqrt{2} Z^+ \quad Z_P \rightarrow \sqrt{2} Z^-$$

Then, the elastic scattering amplitude for positive charged mesons is:

$$\langle + | T | + \rangle = T_1 + (Z^+ Z^+ - Z^- Z^-) T_3 \Rightarrow T_1 - T_3 \quad \text{since } Z^+ | P \rangle = 0$$

Similarly for negative mesons:

$$\langle - | T | - \rangle = T_1 + T_3$$

$$\therefore \frac{\langle + | T | + \rangle + \langle - | T | - \rangle}{2} = T_1 \quad \frac{\langle - | T | - \rangle - \langle + | T | + \rangle}{2} = T_3$$

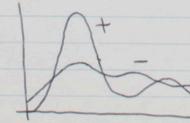
In the laboratory system we know that:

$$\text{Im } \langle + | T | + \rangle = \frac{g}{4\pi} \sigma_T(+)$$

and:

$$\text{Im } \langle - | T | - \rangle = \frac{g}{4\pi} \sigma_T(-)$$

The channel cross sections have the form:



We will have here a logarithmic divergence and we need then make a subtraction. The adequate point to make this subtraction is at zero kinetic energy. Then:

$$\operatorname{Re} T_1 = \operatorname{Re} T_1(\mu) + \frac{q^2}{\pi} P \int \frac{d\omega' 2\omega'}{q'^2(w'^2 - \omega'^2)} \operatorname{Im} T_1$$

$$\operatorname{Re} \left(\frac{T_3}{\omega} \right) = \operatorname{Re} \left(\frac{T_3}{\omega} \right)_{\omega=0} - \frac{q^2}{2\omega^2} \left(\frac{1}{\omega^2} - \frac{1}{\mu^2} \right) + \frac{q^2}{\pi} P \int \frac{d\omega' 2\omega'}{q'^2(w'^2 - \omega'^2)} \operatorname{Im} T_3$$

$$\therefore \operatorname{Re} T_3 = \omega T_3(\mu) + \frac{q^2 \cdot q^2}{2\omega^2 \omega} + \frac{q^2}{\pi} P \int \frac{d\omega' 2\omega'}{q'^2(w'^2 - \omega'^2)} \operatorname{Im} T_3$$

$$\frac{df}{dq} = |f|^2 = \frac{2\pi}{V_c} P(E) \left| \frac{T}{2\omega} \right|^2$$

L stands for laboratory system.

$$\frac{q^2}{\omega} \frac{dq}{d\omega} \frac{1}{(2\pi)^2} \left| \frac{T}{2\omega} \right|^2 = \left| \frac{T}{4\pi} \right|^2 \quad \text{since } \frac{df}{d\omega} = \frac{\omega}{q}$$

$$\therefore f_L = -\frac{T}{4\pi}$$

Then we have:

$$\operatorname{Re} f_1 = \operatorname{Re} f_1(\mu) + \frac{q^2}{\pi} P \int \frac{d\omega' 2\omega'}{q'^2(w'^2 - \omega'^2)} \operatorname{Im} f_1$$

~~$$\operatorname{Re} f_3 = \frac{\omega}{\mu} f_3(\mu) - \frac{q^2 \cdot q^2}{2\omega^2 \omega} + \frac{q^2}{\pi} P \int \frac{d\omega' 2\omega'}{q'^2(w'^2 - \omega'^2)} \operatorname{Im} f_3$$~~

Defining: $\frac{\mu^2 q^2}{4\pi^2} \equiv f^2 \quad \text{where } f^2 \approx 0.08$

the second term of the later equation is: $- \frac{2f^2 q^2}{\mu^2 \omega}$

In order to give the Pomeranchuk argument we need translate our results to C of. M. system. For that we use the next:

$$\frac{f_L}{q_L} = \frac{f_C}{q_C}$$

Note that: $\operatorname{Im} f = \frac{q}{4\pi} \sigma_T \quad \therefore \frac{\operatorname{Im} f}{q} = \frac{\sigma_T}{4\pi}$

We expect that if $\sigma_T \rightarrow \text{constant}$, then $\operatorname{Re} f \rightarrow q$ (also $\operatorname{Im} f \rightarrow q$)

In the C of. M. system:

$$f = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta} \sin^l}{q}$$

For a particle with finite range (say b), since:

$$t \leq r_p \quad \text{for } k < b \quad t \leq b p$$

$\therefore l < b k \quad k = \text{no. of the incident particle}$

We expect only lower values of the angle momentum.

$$\therefore \sum (2l+1) \rightarrow \sum_{l=0}^{\infty} (2l+1) \sim b^2 q^2$$

$$\therefore f \sim b^2 q$$

We have found 1 for the lab. syst.)

$$f_3(\omega) = \omega f_3(\mu) - \frac{2f^2 q^2}{\omega} + \frac{q^2 \omega}{4\pi^2} P \int \frac{d\omega'}{q'^2(w'^2 - \omega'^2)} (\sigma'_- - \sigma'_+)$$

where σ' is the total cross-section (for negative and positive particles respectively). The Pomeranchuk argument is as follows.

$$\sigma \rightarrow A^-$$

$$\sigma \rightarrow A^+$$

$$\sigma \rightarrow A^- + \dots \frac{1}{(\log q)^2} + \dots \sin q$$

For the integral:

$$\begin{aligned} \int \frac{d\omega'}{q'(w'^2 - \omega'^2)} \sigma'_- &= \int \frac{d\omega'}{q'(w'^2 - \omega'^2)} [(\sigma'_- - A_-) + A_-] = \\ &= A_- \int \frac{d\omega'}{q'(w'^2 - \omega'^2)} + \int \frac{d\omega'}{q'(w'^2 - \omega'^2)} (\sigma'_- - A_-) = \\ &= -\frac{A_-}{\omega_2} \int \frac{d\omega'}{q'_1} - \frac{1}{\omega_2} \int \frac{d\omega'}{q'_1} (\sigma'_- - A_-) \end{aligned}$$

The first is only logarithmic divergent and is well known how to handle it. The second goes as $\frac{1}{\log q^2} \sim \frac{d}{dq} \frac{1}{(\log q)^3}$.

Then, the asymptotic behavior of f_3 is:

$$f_3 \rightarrow w a + w b - \frac{w \log \omega}{4\pi^2} (A_- - A_+) \quad (\omega \rightarrow q)$$

$$\therefore \frac{f_3}{q_L} \rightarrow a - (A_- - A_+) \log q_L$$

or: $\operatorname{Re} \frac{f_3}{q_L} \rightarrow a - (A_- - A_+) \log q_L \quad \frac{\operatorname{Im} f_3}{q} \rightarrow (A_- - A_+)$

$$\therefore \operatorname{Re} f_3^L \sim \sum_{l=0}^{bq} (2l+1) \frac{\operatorname{Re} f_L}{q_C} < b^2 q$$

since $l < bq$

Then is reasonable that $f_3(\omega)$ needs no subtraction. The dispersion relations are:

$$\operatorname{Re} f_3(\omega) = \frac{2q^2}{\omega} + \frac{\omega}{4\pi^2} P \int_{-\infty}^{\infty} \frac{(\sigma'_- - \sigma'_+) q' d\omega'}{(q'^2 - q^2)} \quad (\text{without subtraction})$$

and

$$\operatorname{Re} f_1 = f_1(\mu) + \frac{q^2}{4\pi^2} P \int_{-\infty}^{\infty} \frac{(\sigma'_- + \sigma'_+) dq'}{(q'^2 - q^2)}$$

We know, for the forward direction in the positive mesons:

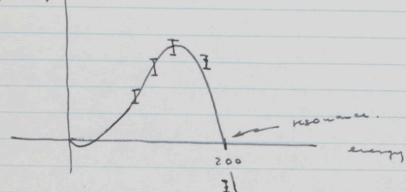
$$\text{Re } \langle + | f | + \rangle = \text{Re } (f_+ - f_0)$$

$$\frac{d\sigma}{d\omega} = |\Delta f|^2 + (\text{Re } f)^2 = q^2 |\sigma_T|^2 + |\text{Re } f|^2$$

$$\therefore \left(\frac{d\sigma}{d\omega} \right)_{\text{forward}} = q^2 |\sigma_T|^2 = |\text{Re } f|^2$$

This is very hard to measure in part by coulomb effects.

$$\text{Re } \langle + | f | + \rangle$$



$$f \sim e^{i\delta} \sin \delta$$

$$\therefore \text{Re } f \sim \cos \delta \sin \delta + f_0$$

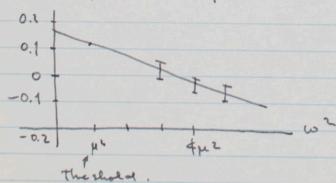
There are very good agreement with the experiments. (with $f^2 = 0.08$)
(Experiments with $+$ mesons were performed by Anderson and with $-$ mesons by
Popov-Puppin. These later are more difficult and Puppin was found a different
value: $f^2 \sim 0.04$) But roughly speaking the agreement is good.
Now, we consider:

$$\text{Re } \omega f_3(\omega) = 2f^2 + \frac{\omega^2}{4\pi^2} \int \frac{q' d\omega' (\rho'_- - \rho'_+)}{\mu} \left[\frac{1}{\omega'_-} + \frac{1}{\omega'_+} \cdot \frac{\omega^2}{\omega'^2 - \omega^2} \right]$$

$$\text{or: } \text{Re } \omega f_3(\omega) - \frac{\omega^4}{4\pi^2} \int \frac{q' d\omega' (\rho'_- - \rho'_+)}{\mu} \frac{\infty}{\omega^2(\omega'^2 - \omega^2)} = 2f^2 + A\omega^2$$

$$\text{where: } A = \frac{1}{4\pi} \int \frac{q' d\omega'}{\omega^2} (\rho'_- - \rho'_+)$$

This has the great advantage to be a straight line.



$$S_{1/2}, S_{3/2}, S_{3/2}, S_{1/2}$$

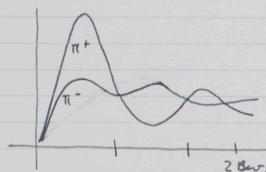
$$S_{1/2} = 0.16 \frac{q}{\mu}$$

$$S_{3/2} = -0.11 \frac{q}{\mu}$$

(This follows the O�ar phase shift analysis. Using the Anderson phases
the agreement is not so good.)

$$f_3 = \frac{t_{1/2} - t_{3/2}}{3}$$

Note that the knowledge of the slope of this line gives the integral A . We will
study that in three regions:



P wave meson states.

This gives the main contribution to scattering and photo production of mesons.
The T-matrix in terms of Dirac functions may be written as: (in general).

$$T = \bar{u}(p_2) [A + \dots] u(p_1)$$

where the bracket contains the invariants that we can construct. From the Dirac
equation:

$$i\gamma \cdot p_2 = i\gamma \cdot p_1 = -m$$

$$p_2 + q_L = p_1 + q_1$$

$$\therefore i\gamma \cdot (q_2 - q_1) - i\gamma \cdot (p_1 - p_2) = 0$$

Then we have: $i\gamma \cdot p_2, i\gamma \cdot p_1, i\gamma \cdot q_2, i\gamma \cdot q_1$

$$\text{or: } i\gamma \Delta q, i\gamma \left(\frac{q_1 + q_2}{2} \right)$$

Then:

$$T = \bar{u}(p_2) [A + i\gamma \cdot \frac{(q_1 + q_2)}{2} B] u(p_1)$$

where A and B are ^{scalar} independent of momenta.

I Breit coord. syst.

$$\bar{u}(-\vec{\Delta}) [(A - \beta \omega B) + i \vec{\gamma} \cdot \vec{Q} B] u(\vec{\Delta})$$

$$\therefore T_1 = A - \omega B$$

$$\text{Since: } \bar{u}(-\Delta) i\gamma u(\Delta) \rightarrow \vec{\Delta} + i \frac{\vec{\gamma} \times \vec{\Delta}}{m}$$

the second term is: $i \frac{\vec{\gamma} \times \vec{\Delta} \cdot \vec{Q}}{m} B$

$$\text{Therefore: } T_2 = \frac{B}{m}$$

Note that not only T_1 and T_2 satisfy dispersion relation but also A and B .
It seems that the disp. rel. for A and B no needs subtraction.

In Born approx: (pseudo scalar theory) For 1st. approx:

$$T_B \sim A_B = 0 \quad B_B = \frac{1}{\omega_0 - \omega} - \frac{1}{\omega_B + \omega}$$

We have for egs

$$\text{Re } T_1 = \text{Born term} + \int \text{Im } T_1 \quad (\text{even function})$$

and similar relation for T_2 (odd), T_3 (odd), T_4 (even)

What will we do to handle them? To do that we use the empirical fact that the resonance peak is very large and substitute the area under the double curve for the area under the resonance. This works very well for 3,3 scattering.

The scattering amplitude for 3,3 shift is:

$$f = \left(\delta\omega_B - \frac{2\omega_2^2 \Delta^2}{3} \right) \left(2\cos\theta - \frac{i \vec{q}_f \times \vec{q}_0}{q_f q_0} \right) f_{33}$$

$$f_{33} = \frac{e^{i\delta_{23}} \sin \delta_{33}}{\hat{f}}$$

$$f(\omega, \Delta^2) = \int \frac{d\omega' \text{Im } f(\omega', \Delta^2)}{\omega'^2 - \omega^2} + \dots$$

and: $\frac{q'^2}{2}(1-\cos\theta') = \Delta^2 \quad \cos\theta' = 1 - \frac{2\Delta^2}{q'^2} = 1 - \frac{2}{q'^2} \left(\frac{q'^2}{2} \right) (1-\cos\theta)$

$$\therefore \cos\theta' = \frac{q'^2}{q'^2} \cos\theta + \left(1 - \frac{q'^2}{q'^2} \right)$$

As result of this theory we have: (Remember the notation $f_{23, 23}$)

$$\text{Re } f_{33} = \frac{4}{3} \frac{f^2 q^2}{\omega} + \frac{q^2}{\pi} P \int \frac{d\omega'}{q'^2} \text{Im } f_{33} \left[\frac{1}{\omega' - \omega} + \frac{1}{9} \frac{1}{\omega' + \omega} \right]$$

$$\text{Re } f_{13} = \text{Re } f_{31} = \frac{\text{Re } f_{11}}{4} = -\frac{2}{3} \frac{f^2 q^2}{\omega} + \frac{q^2}{\pi} P \frac{4}{9} \int \frac{d\omega'}{q'^2} \text{Im } f_{23} \frac{1}{\omega' + \omega}$$

The phase shifts are given by:

$$f = \frac{e^{i\delta} \sin \delta}{\hat{f}}$$